THE MATHEMATICAL GAZETTE

EDITED BY
W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF
F. S. MACAULAY, M.A., D.Sc., F.R.S.
AND
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FUNCTIONS OF AN INFINITE NUMBER OF VARIABLES.

BEING THE SUBSTANCE OF AN ADDRESS GIVEN TO A CONFERENCE OF SECONDARY TEACHERS, HELD AT DURHAM IN FEBRUARY 1927.

BY J. L. BURCHNALL, M.C., M.A.

THE idea of a function of an infinite number of variables, like so many other mathematical ideas, has its origin in the study of problems of applied mathematics. Such problems, as is usual, preceded by some considerable time the explicit formulation of the idea. As an introduction to this discussion it will be sufficient to consider briefly two of them.

The Problem of the Brachistochrone.—Given two points P, Q in a vertical plane, what is the shape of a smooth curve through P, Q for which the time required to descend from P to Q under gravity is a mimimum.

Suppose the equation of the curve to be $y=\phi(x)$.

From the principle of energy $\left(\frac{ds}{dt}\right)^2=2g(y-y_0)$, whence

$$t=\int_{y_0}^{y_1}\frac{(1+\phi'^2)^{\frac{1}{2}}}{\sqrt{2g(\phi-y_0)}}\,dx.$$

We require so to choose ϕ as to make the integral on the right a minimum.

The integral depends upon the *form* of ϕ, that is to say, on the *shape* of the curve, or, if we please, on the value of y for each particular x between x_0 and x_1. Our problem is thus in essence that of determining the critical value of a function of an infinite number of variables.

In this particular instance the solution, which is well known, is obtained by the methods of the Calculus of Variations; but the point to be emphasised in the present connection is that every problem in the Calculus of Variations, depending on the determination of a function form, is in effect a problem involving the determination of an infinite number of variables, to wit, the values of the undetermined function over a certain range of values of its argument.

Our second problem is *Abel's Problem.* With the same data as in the previous problem, find a smooth curve for which the time taken to descend a vertical distance y from P is a *given* function of y.

Let $t=f(y)$,* and suppose the curve determined by its equation in s, y,

$$s=\phi(y).*$$

Then, as before,
$$\left(\frac{ds}{dt}\right)^2=2g(y-y_0);$$

$$t=\int_{y_0}^{y}\frac{\phi'(y)\,dy}{\sqrt{2g(y-y_0)}};$$

* Known functions are throughout denoted by English letters; functions which are analogues of variables by Greek letters.

or, writing $t=f(y)$, $\phi'(y)=\psi(y)$;

$$f(y)=\int_{y_0}^{y}\frac{\psi(y)\,dy}{\sqrt{2g(y-y_0)}}.$$

Here $f(y)$ is known and we seek to determine $\psi(y)$; that is, the shape of the path, or, in other words, an infinite number of unknowns, the values of $\psi(y)$ for each value of y.

Integral Equations.—We have here one of the earliest examples of an *integral equation*, that is, an equation in which the unknown function $\psi(y)$ appears beneath an integral sign. Again we shall not attempt the solution but merely point out that every integral equation requires in a similar manner the determination of an infinity of unknowns.

But here the sceptic may say: "In algebra 'n' equations require for their determination 'n' unknowns. How is it that a single equation suffices to determine an infinity of unknowns?"

The answer to this question serves to introduce a method which has been prolific in providing solutions of integral equations.

For simplicity we take the equation

$$f(y)=\int_{0}^{y}\psi(y)\,dy$$

(which has the advantage that we all know its solution $\psi(y)=f'(y)$), and consider it solely for values of y in the interval (0, 1).

Let the interval (0, 1) be split up into n equal intervals of length $\frac{1}{n}$: on recalling the summation definition of a definite integral we have

$$\int_{0}^{y}=\operatorname*{Lt}_{n\to\infty}\frac{1}{n}\left\{\psi\left(\frac{1}{n}\right)+\psi\left(\frac{2}{n}\right)+\ldots \text{ until we are just up to or past } y\right\}.$$

For the present *let us ignore the limiting sign* and replace y in turn by $\frac{1}{n}, \frac{2}{n} \ldots \frac{n}{n}$ in the integral equation. We thus obtain the set of equations:

$$f\left(\frac{r}{n}\right)=\frac{1}{n}\left\{\psi\left(\frac{1}{n}\right)+\psi\left(\frac{2}{n}\right)+\ldots\psi\left(\frac{r}{n}\right)\right\}, \quad (r=1, 2 \ldots n).$$

We have here n equations in the n unknowns $\psi\left(\frac{r}{n}\right)$ and our paradox is explained. The integral equation is to be regarded as the limiting form of this set of n equations when n is made infinite. If this conclusion is valid, the solution of the integral equation will be found by solving the above set of equations and making n tend to infinity in the solution.

Now by subtracting the $(r-1)$th equation from the rth, we have

$$\frac{1}{n}\psi\left(\frac{r}{n}\right)=f\left(\frac{r}{n}\right)-f\left(\frac{r-1}{n}\right) \text{ for every } r;$$

or, on putting $\quad \frac{r}{n}=y, \quad \frac{1}{n}=h,$

$$\psi(y)=\frac{f(y)-f(y-h)}{h}.$$

If in this result we make h tend to zero, we obtain the solution of the integral equation which we seek:

$$\psi(y)=f'(y).$$

The systematic study of integral equations took in its early days a form of which the example just treated is typical.

A set of algebraical equations in n variables was studied and solved. An integral equation was formed which might be regarded as equivalent to the set of equations when n was made to tend to infinity, and the solution of the

integral equation was deduced by making n tend to infinity in the solution of the algebraical equations. This having been done it was necessary to placate Epsilon by suitable offerings which we have on this occasion omitted.

Analogues of Algebraic Forms.—If such a method is to be effective an important question arises, namely: "What is the appropriate limiting form of a set of algebraic equations of specified type?"

In attempting an answer to this question a diagram will help. Consider first a function of n variables. Take a segment (a, b) of the 'x' axis and divide it into $n-1$ equal parts, erecting n perpendiculars at the ends and at the points of section, and let the values of the n variables $y_1, y_2 \dots y_n$ be set off along these lines as lengths M_1P_1, M_2P_2, $\dots$ M_nP_n. To any position of the broken line $P_1P_2 \dots P_n$ there corresponds a set of values of the variables $y_1, y_2, \dots y_n$, and consequently a value of the function $F(y_1, y_2 \dots y_n)$. The function F is therefore, in a sense, a function of the line since the line determines the values of the variables which in turn determine the function.

The diagram suggests that if we wish to consider functions of an infinite number of variables we should replace the broken line by a curve $y=\phi(x)$

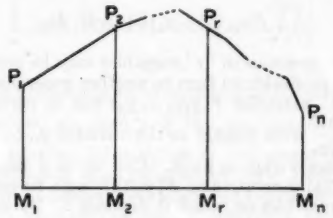

FIG. 1.

and take as our variables all the values of $\phi(x)$ for values of x in (a, b). We may then write appropriately $F[\phi_a{}^b(x)]$ for any quantity which depends upon all these values. The value of $F[\phi_a{}^b(x)]$ will vary with the form of $\phi(x)$, or geometrically with the position of the curve: for this reason the expression is frequently called a "function of a line."

Examples of such a function are not far to seek: the simplest of course is $\int_a^b \phi(x)\,dx$, which, representing the area under the curve, obviously varies with it. Or we may quote from physics the influence of an electric circuit on a magnet, which changes with the form of the circuit.

With the aid of our diagram we may now consider the integral analogues of certain algebraic functions.

The simplest function of n variables is the linear function

$$a_1y_1+a_2y_2+\dots a_ny_n \quad \text{or} \quad \Sigma a_ry_r.$$

The value of the function may be exhibited graphically by drawing a line of length $a_1P_1M_1+a_2P_2M_2+\dots a_nP_nM_n$.

It is thus at once evident that if a finite result is in general to be obtained when n becomes infinite, the multiplying a's must become small in such a way that $\underset{n\to\infty}{\text{Lt}}\, na_r$ is finite.

We therefore prefer to take as the fundamental linear function

$$\frac{b-a}{n}\left\{na_1y_1+\dots+na_ry_r+\dots na_ny_n\right\}=\sum\frac{b-a}{n}na_ry_r.$$

We now construct a fresh diagram on the base (a, b), setting off on 'n' perpendiculars the quantities na_r, each on its appropriate perpendicular, thus obtaining a second broken line $A_1A_2 \ldots A_n$. If now n tends to infinity, the a's increase in number, and the broken line becomes a curve of *known* form $y=f(x)$.

As we have already seen, the y_r's may then be replaced by the values of a function $\phi(x)$. The multiplier $\frac{b-a}{n}$ is appropriately replaced by the differential dx, and on replacing the Σ sign by an integral sign there appears as the analogy of the linear function the integral

$$\int_a^b f(x)\phi(x)\,dx.$$

A quadratic function $\sum_{r=1}^{n}\sum_{s=1}^{n} a_{rs}y_ry_s$ may be generalised in a similar manner.

Since there are now two signs of summation, the appropriate limiting form is a double integral, while the constants having a *double* suffix must be replaced by a known function of *two* variables. The generalisation required is thus:

$$\int_a^b\int_a^b f(x_1, x_2)\phi(x_1)\phi(x_2)\,dx_1\,dx_2.$$

A polynomial homogeneous in 'r' variables may be generalised in a similar way, but it is more profitable to turn to another generalisation.

A function of 'n' variables $F(y_1y_2 \ldots y_n)$ has in certain circumstances a partial derivative $\frac{\partial F}{\partial y_r}$ with respect to the variable y_r, obtained by varying y_r and keeping the other variables fixed. Can we in a similar way attribute a meaning to the partial derivative of $F[\phi_a{}^b(x)]$ with respect to any one of the infinite number of variables on which it depends? We can. We change the form of $\phi(x)$ slightly in the neighbourhood of the ordinate $x=\xi$ which represents the particular variable. The area under the curve $y=\phi(x)$ is thus changed by a small quantity σ. The change in the form of the curve will cause also a small change in the value of $F[\phi_a{}^b(x)]$ which we may denote by $\delta F[\phi_a{}^b(x)]$. In a large number of cases the ratio $\frac{\delta F}{\sigma}$ tends to a limit when, with certain restrictions as to shape, the variation in the form of $y=\phi(x)$ tends to zero. We have thus a generalisation of the idea of a partial derivative, and we may write the limiting value as

$$F'_\xi[\phi_a{}^b(x)].$$

With such a definition of a derivative we may establish that:

If $$F[\phi_a{}^b(x)]=\int_a^b f(x)\phi(x)\,dx, \quad \text{then} \quad F'_\xi[\phi_a{}^b(x)]=f(\xi),$$

and if $$F[\phi_a{}^b(x)]=\int_a^b\int_a^b f(x_1x_2)\phi(x_1)\phi(x_2)\,dx_1\,dx_2,$$

then $$F'_\xi[\phi_a{}^b(x)]=\int_a^b f(x_1\xi)\phi(x_1)\,dx_1.$$

These are evident analogies of the fact that a derivative of a linear function is a constant while the derivative of a quadratic function is a linear function.

It is of course possible to define in a similar manner the analogues of derivatives of higher order, and, armed with these, to proceed to results which are analogous to Taylor's and Maclaurin's theorems in ordinary analysis.

Generalisation of Distance.—The distance[2] between points (y_1, y_2), (y_1', y_2') in two dimensions is given by $d^2=(y_1-y_1')^2+(y_2-y_2')^2$.

In three dimensions we have a similar formula:

$$d^2=(y_1-y_1')^2+(y_2-y_2')^2+(y_3-y_3')^2;$$

and in n dimensions distance is defined by

$$d^2 = \sum_{r=1}^{n} (y_r - y_r')^2.$$

If in the manner of a previous paragraph we choose to represent the points in 'n' dimensions by broken lines:

$$P_1P_2 \dots P_n, \quad P_1'P_2' \dots P_n',$$

then

$$\frac{b-a}{n} \sum_{r=1}^{n} (y_r - y_r')^2$$

provides a measure of the deviation of one broken line from the other.

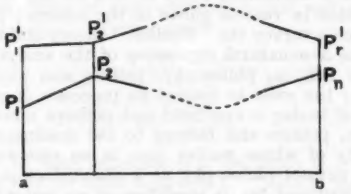

FIG. 2.

We are thus led by proceeding to an infinite number of variables to consider $\int_a^b [f(x)-g(x)]^2 dx$, which we may call the mean square of the distance between the curve $y=f(x)$, $y=g(x)$, or if we please between the points $f(x)$, $g(x)$ in functional space of an infinite number of dimensions.

With this definition of distance between two functions we may proceed to define what we mean by "neighbouring" functions, and develop from this basis a theory of continuity in the functional field.

Application to Approximation.—The idea has however its practical uses.

Suppose that we have a function $f(x)$ defined in $(0, \pi)$ to which we seek an approximation $g(x)$ in the form of a cosine series limited to $(n+1)$ terms:

$$g(x) = b_0 + b_1 \cos x + b_2 \cos 2x + \dots + b_n \cos nx.$$

From our present point of view * it is natural to regard that approximation as the best which makes $\int_0^\pi [f(x)-g(x)]^2 dx$ as small as possible, the 'b_r's being chosen to effect this. Regarding the integral as a function of the 'b_r's a little simple analysis shows at once that the best possible approximation will be attained when the 'b_r's are the well known Fourier coefficients.

Further Illustrations of the General Idea.—The idea of a function of an infinite number of variables is of more common occurrence than might be supposed. I illustrate its occurrence by a statement of two further problems.

(*a*) The differential equation

$$\frac{d^2y}{dx^2} + \phi(x)y = 0$$

has under certain conditions one solution y for which $y=0$, $y'=1$ when $x=0$.

This solution depends upon the *form* of $\phi(x)$, that is, on all the values $\phi(x)$ assumes for relevant values of x. It is thus a function of an infinite number of variables. We may ask what is the effect upon the solution of a variation in the form of $\phi(x)$. Researches on this and analogous problems date from the time of Sturm and have led to a remarkable set of theorems. The ideas involved are essentially those we have discussed.

* The method has evident analogies with the method of Least Squares.

(*b*) The solution of the harmonic equation

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0$$

is determined throughout the *inside* of a closed surface S if we know the values which V takes *on* the surface. To give V at every point on S is equivalent to giving an infinite number of values. The solution required depends upon all these values, and so is a function of an infinite number of parameters. It is not therefore surprising that modern discussion of boundary problems is to a considerable extent based upon the ideas I have endeavoured to place before you.

Conclusion.—It is a fascinating study to seek for the dominant thought of an epoch as exhibited in various guises in the different branches of human knowledge. In such a survey the "Epsilon" theory and the "Mathematical Logic" of our time is a natural expression of the analytic spirit of an age which, alike in its religion, philosophy, politics and physics, is constantly digging itself up by the roots to inspect its processes of growth. But there is in the thought of to-day a synthetic and perhaps more significant trend. It leads in religion, politics and biology to the dominant conception of an organism, an entity of whose reality time is an essential constituent. It expresses itself in natural philosophy as a dissatisfaction with the classical view of nature determined by its condition at an instant and a set of differential equations.

Even where this dissatisfaction is not felt there is still the realisation that in many problems where the conditions at an instant are not and cannot through their complexity be known, an organic hypothesis is the more fruitful.

Is it fanciful to see in the integral equation and allied theory at once another variant of this mode of thought and a weapon by which, if at all, the problems of the organic universe must be attacked by mathematicians?

J. L. BURCHNALL.

GLEANINGS FAR AND NEAR.

685. GEOMETRY AND ANALYSIS.—Ce Théorème mérite particulièrement l'attention des Géomètres par lui-même, et parce qu'il paraît difficile d'y parvenir par le calcul; en sorte qu'on pourrait le mettre dans le petit nombre de ceux pour lesquels l'Analyse géométrique semble avoir de l'avantage sur l'Analyse algébrique.

Lagrange on Lambert's Theorem (*œuvres*, vol. 4, p. 560; reprinted from *Nouv. Mém. de l'Acad. . . . de Berlin*, 1778). On which Sylvester comments (*Math. Papers*, vol. 2, p. 521; reprinted from *Phil. Mag.* 1866):

In the nature of things such advantage can never be otherwise than temporary. Geometry may sometimes appear to take the lead of analysis, but in fact precedes it only as a servant goes before his master to clear the path and light him on his way. The interval between the two is as wide as between empiricism and science, as between the understanding and the reason, or as between the finite and the infinite.

[It is to be remembered that "géomètre" in French is no narrower than "mathematician" in English; Lagrange's choice of this word has no bearing on the point which he raises. Sylvester's paper *Astronomical Prolusions*, in which his comment occurs, was reprinted separately (a copy bound in limp morocco passed through Sotheran's hands some years ago), and also with a few other papers under the title of *Nugæ Mathematicæ*.] [Per Prof. E. H. Neville.]

686. Airy told me—when I was an undergraduate—that no demonstration ever persuaded him that $x-x^3/(2\,.\,3)+x^5/(2\,.\,3\,.\,4\,.\,5)-\ldots$ was periodic in value, until he tried it on numbers of some value, as $x=10$.—De Morgan (in letter to Sylvester, 14th May, 1865).

INTRODUCTION TO ELLIPTIC FUNCTIONS.

By N. M. Gibbins, M.A.

1. If we proceed to find the length of an arc of the circle of unit radius, whose equation is $y^2=1-x^2$, we obtain, if σ is the length required,

$$\left(\frac{d\sigma}{dx}\right)^2=1+\left(\frac{dy}{dx}\right)^2=1+\frac{x^2}{1-x^2}=\frac{1}{1-x^2},$$

so that
$$\frac{dx}{d\sigma}=\sqrt{1-x^2}. \tag{1}$$

Similarly for an ellipse of unit semi-major axis and eccentricity e, whose equation is $y^2=(1-e^2)(1-x^2)$, we obtain

$$\left(\frac{d\sigma}{dx}\right)^2=1+\frac{(1-e^2)x^2}{1-x^2}=\frac{1-e^2x^2}{1-x^2};$$

or
$$\frac{dx}{d\sigma}=\sqrt{\frac{1-x^2}{1-e^2x^2}}. \tag{2}$$

We know how to solve equation (1), the result is $x=\sin(\sigma+c)$; but we do not know yet how to solve equation (2). But if we did not happen to know the solution of equation (1), the following reasoning will show how to find out all about it, if it exists; and then the same method will be used to solve equation (2), or, at any rate, a modification of it.

2. Changing σ into u, and x into $\sin u$, in equation (1), we define the function $\sin u$ by the relations

$$\frac{d}{du}(\sin u)=\sqrt{1-\sin^2 u}, \tag{3}$$

$$\sin 0=0. \tag{4}$$

But since the radical is going to make great difficulties in calculation, we introduce a new function called $\cos u$, which is such that

$$\sqrt{1-\sin^2 u}=\cos u, \quad \text{or} \quad \cos^2 u+\sin^2 u=1.$$

We have then for our definitions:

$$\frac{d}{du}(\sin u)=\cos u, \tag{5}$$

$$\cos^2 u+\sin^2 u=1, \tag{6}$$

$$\sin 0=0, \tag{4}$$

$$\cos 0=1. \tag{7}$$

The equations of definition are also satisfied if we change the signs of u and of $\sin u$.

Thus
$$\sin(-u)=-\sin u, \tag{8}$$

and
$$\cos(-u)=\cos u. \tag{9}$$

Calling $\sin u$ and $\cos u$, s and c, for short; and denoting differentiation by dashes, we have

$$s'=c, \tag{10}$$

and
$$c^2+s^2=1. \tag{11}$$

3. *Expansion in series.*

Differentiating equation (11) we get $cc'+ss'=0$, or $cc'+sc=0$; and since c is not generally equal to zero, we have

$$c'=-s. \tag{12}$$

Differentiating equation (10) $s''=c'=-s$ by (12),
" " this $s'''=-s'=-c$ by (10),
" " " $s''''=-c'=s$ by (12),
and so on; the set $c, -s, -c, s$ being always repeated in that order.

Putting $u=0$ in all these results and applying Maclaurin's theorem, we obtain

$$\sin u = u - \frac{u^3}{\lfloor 3} + \frac{u^5}{\lfloor 5} - \text{etc.}, \tag{13}$$

and

$$\cos u = 1 - \frac{u^2}{\lfloor 2} + \frac{u^4}{\lfloor 4} - \text{etc.} \tag{14}$$

4. *Addition formulae.*

To find $\sin(u_1+u_2)$, put $u_1+u_2=c$, a constant. Also write s_1 for $\sin u_1$, etc. Since $u_1+u_2=u_2+u_1$, if $\sin(u_1+u_2)$ is to be expressed in terms of s_1, s_2, c_1, c_2, only symmetrical expressions in these quantities need be considered; in particular s_1+s_2 and c_1+c_2. Since $du_2=-du$, we have

$$\frac{d}{du_1}(s_1+s_2)=c_1-c_2 \tag{15}$$

and

$$\frac{d}{du_1}(c_1+c_2)=-s_1+s_2. \tag{16}$$

Also, since $c_1^2+s_1^2=1=c_2^2+s_2^2$;

$$\therefore\ c_1^2-c_2^2=-s_1^2+s_2^2. \tag{17}$$

Hence from equations (15), (16) and (17):

$$(c_1+c_2)\frac{d}{du_1}(s_1+s_2)=(s_1+s_2)\frac{d}{du_1}(c_1+c_2),$$

so that

$$\frac{s_1+s_2}{c_1+c_2}=\text{a const.}=\frac{\sin(u_1+u_2)}{\cos(u_1+u_2)+1} \tag{18}$$

on writing u_1+u_2 for u_1 and 0 for u_2.

Again, from equations (17) and (18):

$$\frac{-s_1+s_2}{c_1-c_2}=\text{a constant also}=\frac{\sin(u_1+u_2)}{1-\cos(u_1+u_2)} \tag{19}$$

on making the same substitutions.

Adding the reciprocals of equations (18) and (19):

$$\frac{2}{\sin(u_1+u_2)}=\frac{c_1+c_2}{s_1+s_2}+\frac{c_1-c_2}{-s_1+s_2}=\frac{2(s_2c_1-s_1c_2)}{s_2^2-s_1^2}.$$

But $s_2^2-s_1^2=s_2^2(s_1^2+c_1^2)-s_1^2(s_2^2+c_2^2)=s_2^2c_1^2-s_1c_2^2.$

Hence, finally,

$$\sin(u_1+u_2)=s_1c_2+s_2c_1. \tag{20}$$

Changing u_2 into $-u_2$ we have also from equations (8) and (9):

$$\sin(u_1-u_2)=s_1c_2-s_2c_1. \tag{21}$$

Similarly we may prove that

$$\cos(u_1\pm u_2)=c_1c_2\mp s_1s_2. \tag{22}$$

5. *Periodicity.*

The equations of definition (5) and (6) are satisfied if $\cos u$ is put for $\sin u$ and $-\sin u$ for $\cos u$. But $\cos 0=1$; hence if α is such that $\sin\alpha=1$, $\cos\alpha=0$, we shall have

$$\cos u=\sin(u+\alpha) \quad \text{and} \quad -\sin u=\cos(u+\alpha).$$

Hence

$$\sin(u+2\alpha)=\cos(u+\alpha)=-\sin u,$$
$$\cos(u+2\alpha)=-\sin(u+\alpha)=-\cos u,$$
$$\sin(u+4\alpha)=-\sin(u+2\alpha)=\sin u,$$
$$\cos(u+4\alpha)=-\cos(u+2\alpha)=\cos u.$$

Hence the functions are unchanged when u is increased by 4α, where $\sin\alpha=1$, *i.e.*, they are said to be periodic functions.

It remains to find α.

In the usual way we obtain from equation (20)

$$\sin\alpha=3\sin\frac{\alpha}{3}-4\sin^3\frac{\alpha}{3}.$$

Let $\frac{\alpha}{3}=x$; then since $\sin\alpha=1$,

$$1=3\sin x-4\sin^3 x,$$

or
$$(2\sin x-1)^2(\sin x+1)=0,$$

so that
$$\sin x=\tfrac{1}{2},$$

or
$$x-\frac{x^3}{6}+\frac{x^5}{120}-\ldots=\frac{1}{2} \text{ from equation (13).}$$

Taking the first term only, $x=\frac{1}{2}$.

Taking the first two terms, putting $x=\frac{1}{2}+y$, expanding and rejecting the terms in y^2 and y^3, we have

$$\tfrac{1}{2}+y-\tfrac{1}{6}(\tfrac{1}{8}+\tfrac{3}{4}y)=\tfrac{1}{2},$$

whence
$$y=\tfrac{1}{12}.$$

Hence $x=\frac{1}{2}+\frac{1}{12}$, and $6x$ or $2\alpha=3\frac{1}{2}$, the familiar approximation for π.

Also from equation (1) or (3) we have, since $\sin\frac{\alpha}{3}=\frac{1}{2}$,

$$\begin{aligned}\frac{\alpha}{3}&=\int_0^{\frac{1}{2}}\frac{dx}{\sqrt{1-x^3}}\\&=\int_0^{\frac{1}{2}}dx\left[1+\frac{1}{2}x^3+\frac{1.3}{2.4}x^4+\frac{1.3.5}{2.4.6}x^6+\ldots\right]\\&=\left(\frac{1}{2}\right)+\frac{1}{2}\cdot\frac{1}{3}\left(\frac{1}{2}\right)^3+\frac{1.3}{2.4}\cdot\frac{1}{5}\left(\frac{1}{2}\right)^5+\frac{1.3.5}{2.4.6}\cdot\frac{1}{7}\left(\frac{1}{2}\right)^7+\ldots\end{aligned}\qquad(23)$$

so that α or $\frac{\pi}{2}$ can be found to any degree of accuracy required.

The important thing to note is that $\frac{\pi}{6}$ is found as the above convergent series, and is therefore a definite number.

6. *Length of the circumference of a circle.*

From equation (1) the length of the arc of the circle, from $x=0$ to $x=x_1$, is $\int_0^{x_1}\frac{dx}{\sqrt{1-x^3}}$, and on putting $x=\sin u$, the result is u_1, where $x_1=\sin u_1$. If the quarter circumference is required, x varies through a series of values beginning with 0 and ending with 1, while the value of u varies from 0 to α or $\frac{\pi}{2}$. The circumference of a circle is thus 2π times its radius, where π is obtained from equation (23), and this result has been obtained without reference to any geometrical meaning that $\sin u$ and $\cos u$ may have. Everything has been deduced from equation (1), the practical problem proposed leading to the invention of the functions needed to solve it.

7. It is true that the new functions which we are now going to investigate got their name "elliptic" functions from their connection with the length of an arc of an ellipse, but equation (2) is not the one which occurs most frequently

in the solution of problems. In the case of the pendulum, for example, the equation which arises out of the dynamics of its mode of swing is of the form :

$$\frac{dx}{dt}=\sqrt{1-x^2}\times\sqrt{1-k^2x^2} \text{ where } k<1, \tag{24}$$

and this is the equation on which the theory of elliptic functions is based.

Avoiding radicals as before, the functions called $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$ are defined as follows, u being called the *argument*:

$$\frac{d}{du}(\operatorname{sn} u)=\operatorname{cn} u \operatorname{dn} u, \tag{25}$$

$$\operatorname{cn}^2 u+\operatorname{sn}^2 u=1, \tag{26}$$

$$\operatorname{dn}^2 u+k^2\operatorname{sn}^2 u=1, \quad k<1, \tag{27}$$

$$\operatorname{sn} 0=0, \tag{28}$$

$$\operatorname{cn} 0=1, \tag{19}$$

$$\operatorname{dn} 0=1. \tag{30}$$

Equations (25), (26) and (27) are between them equivalent to equation (24), just as equations (5) and (6) were equivalent to equation (3).

In the development of the theory of elliptic functions from the above equations of definition, it is important to realise at the outset that the only difficulty is that the algebra is much heavier and longer, while the method is exactly the same as in the preceding work.

Differentiating equation (26), we have

$$\operatorname{cn} u\frac{d}{du}(\operatorname{cn} u)+\operatorname{sn} u\frac{d}{du}(\operatorname{sn} u)=0;$$

$$\therefore \operatorname{cn} u\frac{d}{du}(\operatorname{cn} u)+\operatorname{sn} u \operatorname{cn} u \operatorname{dn} u \text{ from equation (25)},$$

and since $\operatorname{cn} u$ is not zero in general,

$$\frac{d}{du}(\operatorname{cn} u)=-\operatorname{sn} u \operatorname{dn} u. \tag{31}$$

Differentiating equation (27), we have

$$\operatorname{dn} u\frac{d}{du}(\operatorname{dn} u)+k^2\operatorname{sn} u\frac{d}{du}(\operatorname{sn} u)=0,$$

and

$$\therefore \frac{d}{du}(\operatorname{dn} u)=-k^2\operatorname{sn} u \operatorname{cn} u. \tag{32}$$

The equations of definition are also satisfied if we change the signs of u and of $\operatorname{sn} u$.

Hence

$$\operatorname{sn}(-u)=-\operatorname{sn} u, \tag{33}$$

$$\operatorname{cn}(-u)=\operatorname{cn} u, \tag{34}$$

$$\operatorname{dn}(-u)=\operatorname{dn} u. \tag{35}$$

9. *Addition formulae.*

As before, put $u_1+u_2=c$, so that $du_2=-du_1$, and let $\operatorname{sn} u_1=s_1$, $\operatorname{cn} u_1=c_1$, $\operatorname{dn} u_1=d_1$, etc.

Considering only symmetric functions of u_1 and u_2, such as $s_1c_2+s_2c_1$, d_1+d_2, we have

$$\frac{d}{du_1}(s_1c_2+s_2c_1)=c_1d_1c_2-s_1d_1s_2+s_1s_2d_2-c_2d_2c_1 \text{ from equations (25) and (31)},$$

$$=(d_1-d_2)(c_1c_2-s_1s_2), \tag{36}$$

$$\frac{d}{du_1}(d_1+d_2)=-k^2(s_1c_1-s_2c_2) \text{ from equation (32)},$$

$$=-k^2s_1c_1(c_2^2+s_2^2)+k^2s_2c_2(c_1^2+s_1^2)$$

$$=-k^2(c_1c_2-s_1s_2)(s_1c_2-s_2c_1). \tag{37}$$

Now $$-k^2(s_1^2c_2^2-s_2^2c_1^2)=-k^2(s_1^2-s_2^2)=d_1^2-d_2^2 \text{ by equation (27).} \tag{38}$$

Hence from equations (36), (37) and (38):

$$(d_1+d_2)\frac{d}{du_1}(s_1c_2+s_2c_1)=(s_1c_2+s_2c_1)\frac{d}{du_1}(d_1+d_2);$$

and
$$\therefore\ \frac{s_1c_2+s_2c_1}{d_1+d_2}=\text{a constant}=\frac{\operatorname{sn}(u_1+u_2)}{1+\operatorname{dn}(u_1+u_2)}, \tag{39}$$

on writing u_1+u_2 for u_1 and 0 for u_2.

Again, from equations (38) and (39):

$$\frac{s_1c_2-s_2c_1}{d_1-d_2}=-\frac{1}{k^2}\cdot\frac{d_1+d_2}{s_1c_2+s_2c_1}=\text{a constant}=\frac{\operatorname{sn}(u_1+u_2)}{\operatorname{dn}(u_1+u_2)-1}. \tag{40}$$

Subtracting the reciprocals of equations (39) and (40), we get

$$\frac{2}{\operatorname{sn}(u_1+u_2)}=\frac{d_1+d_2}{s_1c_2+s_2c_1}-\frac{d_1-d_2}{s_1c_2-s_2c_1}=\frac{2(s_1c_2d_2-2s_2c_1d_1)}{s_1^2c_2^2-s_2^2c_1^2}$$

so that
$$\operatorname{sn}(u_1+u_2)=\frac{s_1^2-s_2^2}{s_1c_2d_2-s_2c_1d_1}. \tag{41}$$

But
$$s_1^2c_2^2d_2^2-s_2^2c_1^2d_1^2$$
$$=s_1^2(1-s_2^2)(1-k^2s_2^2)-s_2^2(1-s_1^2)(1-k^2s_1^2)$$
$$=(s_1^2-s_2^2)(1-k^2s_1^2s_2^2). \tag{42}$$

Hence, finally,
$$\operatorname{sn}(u_1+u_2)=\frac{s_1c_2d_2+s_2c_1d_1}{1-k^2s_1^2s_2^2}. \tag{43}$$

Adding the reciprocals of equations (39) and (40), and using equation (41), we obtain

$$\operatorname{dn}(u_1+u_2)=\frac{s_1c_2d_1-s_2c_1d_2}{s_1c_2d_2-s_2c_1d_1}.$$

Now
$$(s_1c_2d_1-s_2c_1d_2)(s_1c_2d_2+s_2c_1d_1)$$
$$=d_1d_2(s_1^2c_2^2-s_2^2c_1^2)-s_1s_2c_1c_2(d_2^2-d_1^2)$$
$$=(s_1^2-s_2^2)(d_1d_2-k^2s_1s_2c_1c_2).$$

Hence using equation (42):

$$\operatorname{dn}(u_1+u_2)=\frac{d_1d_2-k^2s_1s_2c_1c_2}{1-k^2s_1^2s_2^2}. \tag{44}$$

By differentiating c_1+c_2 and $s_1d_2+s_2d_1$, we obtain

$$\frac{s_1d_2+s_2d_1}{c_1+c_2}=\frac{\operatorname{sn}(u_1+u_2)}{\operatorname{cn}(u_1+u_2)+1}.$$

Whence using equations (41) and (42), we obtain

$$\operatorname{cn}(u_1+u_2)=\frac{s_1c_1d_2-s_2c_2d_1}{s_1c_2d_2-s_2c_1d_1}=\frac{c_1c_2-s_1s_2d_1d_2}{1-k^2s_1^2s_2^2}. \tag{45}$$

Changing u_2 into $-u_2$ we get the elliptic functions of u_1-u_2, the results being as in equations (43), (44) and (45) with the sign between the terms of the numerator changed in each case.

It is worth noting that if in these equations we put $k=0$, so that $\operatorname{dn}u_1=1$, etc., we get the addition formulae (20), (21), (22) for the circular functions, as is to be expected by comparing equations (25) and (26) with equations (5) and (6).

Putting $u_2=u_1\equiv u$ in equations (43), (44) and (45), we obtain

$$\operatorname{sn}2u=2sc\,d/(1-k^2s^4),$$

$$\operatorname{dn}2u=\frac{d^2-k^2s^2c^2}{1-k^2s^4}=\frac{1-k^2s^2-k^2s^2(1-s^2)}{1-k^2s^4}=\frac{1-2k^2s^2+k^2s^4}{1-k^2s^4},$$

and
$$\operatorname{cn}2u=\frac{c^2-s^2d^2}{1-k^2s^4}=\frac{1-s^2-s^2(1-k^2s^2)}{1-k^2s^4}=\frac{1-2s^2+k^2s^4}{1-k^2s^4}.$$

Also $1+\mathrm{dn}\,2u=2(1-k^2s^2)/(1-k^2s^4)$,

and $1-\mathrm{cn}\,2u=2s^2(1-k^2s^2)/(1-k^2s^4)$.

Hence $$\mathrm{sn}\,u=\left\{\frac{1-\mathrm{cn}\,2u}{1+\mathrm{dn}\,2u}\right\}^{\frac{1}{2}},$$

or changing u into $\frac{1}{2}u$,

$$\mathrm{sn}\,\tfrac{1}{2}u=\left\{\frac{1-\mathrm{cn}\,u}{1+\mathrm{dn}\,u}\right\}^{\frac{1}{2}}. \tag{46}$$

Similarly we can find $\mathrm{sn}\,3u$, etc., and indeed a complete theory of multiple and sub-multiple arguments can be built up similar to that in ordinary trigonometry.

10. *Periodicity.*

In the former case we found the period α from the equation

$$\frac{d}{du}(\cos u)=-\sin u,$$

and we must now derive similar equations for the new functions.

We have

$$\frac{d}{du}\left(\frac{c}{d}\right)=\frac{d(-sd)-c(-k^2sc)}{d^2} \text{ from equations (31) and (32),}$$

$$=\frac{-s(1-k^2s^2)+k^2s(1-s^2)}{d^2}$$

$$=-\frac{(1-k^2)s}{d^2}=\left(-\frac{k's}{d}\right)\left(\frac{k'}{d}\right) \text{ where } k'^2=1-k^2.$$

Also $$1-\left(\frac{c}{d}\right)^2=\frac{d^2-c^2}{d^2}=\frac{1-k^2s^2-1+s^2}{d^2}=\frac{k'^2s^2}{d^2}=\left(-\frac{k's}{d}\right)^2,$$

and $$1-k^2\left(\frac{c}{d}\right)^2=\frac{1-k^2s^2-k^2(1-s^2)}{d^2}=\frac{k'^2}{d^2}=\left(\frac{k'}{d}\right)^2.$$

Hence $\left(\frac{c}{d}\right)$, $\left(-\frac{k's}{d}\right)$, $\left(\frac{k'}{d}\right)$ satisfy the equations of definitions (25), (26), (27). But now the value of $\left(\frac{c}{d}\right)$ when $u=0$ is 1. Let then K be such a value of u that

$$\mathrm{sn}\,K=1, \text{ so that } \mathrm{cn}\,K=0 \text{ and } \mathrm{dn}\,K=k'. \tag{47}$$

As before we shall have

$$\mathrm{sn}(u+K)=\frac{c}{d}, \tag{48}$$

$$\mathrm{cn}(u+K)=-\frac{k's}{d}, \tag{49}$$

$$\mathrm{dn}(u+K)=\frac{k'}{d}. \tag{50}$$

We have also $$\mathrm{sn}(u+2K)=\frac{\mathrm{cn}(u+K)}{\mathrm{dn}(u+K)}=-\mathrm{sn}\,u,$$

$$\mathrm{cn}(u+2K)=-\frac{k'\,\mathrm{sn}(u+K)}{\mathrm{dn}(u+K)}=-\mathrm{cn}\,u,$$

$$\mathrm{dn}(u+2K)=\frac{k'}{\mathrm{dn}(u+K)}=\mathrm{dn}\,u.$$

Hence $$\mathrm{sn}(u+4K)=-\mathrm{sn}(u+2K)=\mathrm{sn}\,u,$$

$$\mathrm{cn}(u+4K)=-\mathrm{cn}(u+2K)=\mathrm{cn}\,u,$$

$$\mathrm{dn}(u+4K)=\mathrm{dn}(u+2K)=\mathrm{dn}\,u.$$

Hence all three functions have the period $4K$.

From equations (24) and (25) we have, since $\operatorname{sn} K=1$,

$$K=\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}=\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}, \text{ where } x=\sin\theta,$$

$$=\int_0^{\frac{\pi}{2}} d\theta\left[1+\frac{1}{2}k^2\sin^2\theta+\frac{1.3}{2.4}k^4\sin^4\theta+\frac{1.3.5}{2.4.6}k^6\sin^6\theta\ldots\right]$$

$$=\frac{\pi}{2}\left[1+\left(\frac{1}{2}\right)^2k^2+\left(\frac{1.3}{2.4}\right)^2k^4+\left(\frac{1.3.5}{2.4.6}\right)^2k^6+\ldots\right]$$

on substituting the values of $\int_0^{\frac{\pi}{2}} \sin^{2m}\theta\, d\theta$.

Compare this equation (23) where α was found as an infinite series.

We now proceed to show that the elliptic functions have another period.

We have $$\frac{d}{du}\left(\frac{1}{ks}\right)=-\frac{cd}{ks^2}=\left(\frac{id}{ks}\right)\left(\frac{ic}{s}\right).$$

Also $$1-\left(\frac{1}{ks}\right)^2=-\frac{d^2}{k^2s^2}=\left(\frac{id}{ks}\right)^2,$$

and $$1-k^2\left(\frac{1}{ks}\right)^2=-\frac{c^2}{s^2}=\left(\frac{ic}{s}\right)^2.$$

Hence $\frac{1}{ks}$, $\frac{id}{ks}$, $\frac{ic}{s}$ satisfy the equations of definition, but the values are all infinite when $u=0$. The ratio of the second to the first is, however, id, which is equal to i when $u=0$.

Hence, if L is such that $\operatorname{cn} L/\operatorname{sn} L=i$, we obtain

$$\operatorname{sn}(u+L)=\frac{1}{ks},$$

$$\operatorname{cn}(u+L)=\frac{id}{ks},$$

$$\operatorname{dn}(u+L)=\frac{ic}{s}.$$

Hence also $$\operatorname{sn}(u+2L)=\frac{1}{k\operatorname{sn}(u+L)}=\operatorname{sn}u,$$

$$\operatorname{cn}(u+2L)=\frac{i\operatorname{dn}(u+L)}{k\operatorname{sn}(u+L)}=-\operatorname{cn}u,$$

$$\operatorname{dn}(u+2L)=\frac{i\operatorname{cn}(u+L)}{\operatorname{sn}(u+L)}=-\operatorname{dn}u;$$

and $$\operatorname{sn}(u+4L)=+\operatorname{sn}(u+2L)=\operatorname{sn}u,$$

$$\operatorname{cn}(u+4L)=-\operatorname{cn}(u+2L)=\operatorname{cn}u,$$

$$\operatorname{dn}(u+4L)=-\operatorname{dn}(u+2L)=\operatorname{dn}u.$$

The functions have therefore the period $4L$ as well as $4K$.

To evaluate L we note that $\operatorname{sn}(K+L)=\frac{1}{k\operatorname{sn}K}=\frac{1}{k}$.

Hence $$L=\int_1^{\frac{1}{k}} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$$

$$=\int_k^1 \frac{dy}{\sqrt{(y^2-1)(y^2-k^2)}}, \text{ where } x=\frac{1}{y}$$

$$= \int_1^0 \frac{-k'^2 z\,dz}{y} \cdot \frac{1}{k'\sqrt{1-z^2}} \cdot \frac{1}{k'i}, \text{ where } y^2 = 1 - k'^2 z^2$$

$$= \frac{1}{i}\int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-k'^2z^2)}} = \frac{K'}{i},$$

where K' is formed from K by substituting k' for k, where $k'^2 + k^2 = 1$.

11. *Geometrical illustrations.*

Take any four points A, A', B', B in order on a straight line, and on AB, $A'B'$ as diameters describe circles. Let the centres be O, O', the radii R, r, and let $OO' = \delta$.

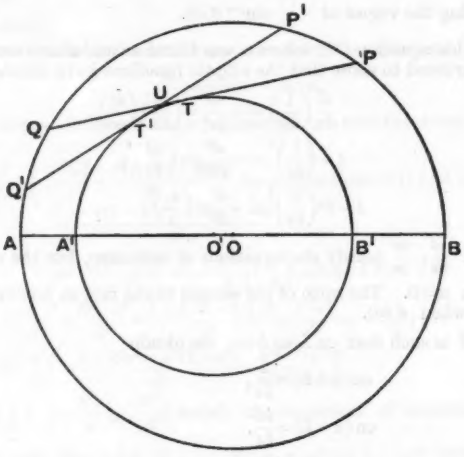

FIG. 1.

Let P, Q be two points on the outer circle, such that PQ touches the inner circle at T. Let $P'T'Q'$ be a consecutive position of PTQ, meeting it in U.

Also let $\theta = BAP$, $\phi = BAQ$, $\theta + d\theta = BAP'$, $\phi + d\phi = BAQ'$.

Then $BOP = 2\theta$, $POP' = 2d\theta$, $PP' = 2Rd\theta$, $QQ' = 2Rd\phi$.

The triangles $PP'U$, $Q'QU$ are similar, hence $PP'/PU = QQ'/UQ'$, and in the limit

$$d\theta/PT = d\phi/TQ. \tag{51}$$

Again $PT^2 = OP^2 - OT^2 = R^2 + \delta^2 + 2R\delta\cos 2\theta - r^2$

$$= (R+\delta)^2 - r^2 - 4R\delta\sin^2\theta$$

and $$TQ^2 = (R+\delta)^2 - r^2 - 4R\delta\sin^2\phi.$$

Now let $$k^2 = 4R\delta/\{(R+\delta)^2 - r^2\}, \tag{52}$$

$$\sin\theta = \operatorname{sn} u,$$

and $$\sin\phi = \operatorname{sn}\phi.$$

Then $$\cos\theta = \operatorname{cn} u,$$

$$\cos\phi = \operatorname{cn}\phi,$$

$$PT = \{(R+\delta)^2 - r^2\}^{\frac{1}{2}} \operatorname{dn} u, \tag{53}$$

$$TQ = \{(R+\delta)^2 - r^2\}^{\frac{1}{2}} \operatorname{dn} v. \tag{54}$$

Also $\cos\theta\, d\theta = \operatorname{cn} u \operatorname{dn} u\ du,$

or $d\theta = \operatorname{dn} u\ du,$

and $d\phi = \operatorname{dn} v\ dv.$

Hence combining these equations with (51), (53) and (54), we obtain

$$du = dv,$$

or $v - u = a$, a constant.

To find a, take the position BQ_0 of the chord PQ.

Then at B, $u = 0$; and at Q_0, $v = \sigma$,

and
$$\operatorname{cn} a = \cos\phi_0 = \left(\frac{r}{R+\delta}\right) \tag{55}$$

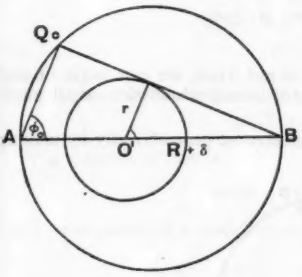

FIG. 2.

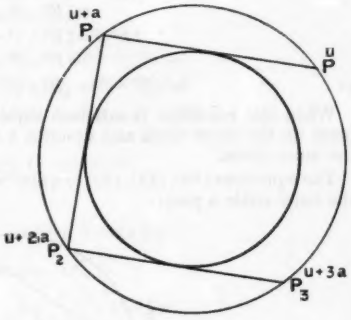

FIG. 3.

Further
$$\operatorname{sn}^2 a = 1 - \left(\frac{r}{R+\delta}\right)^2,$$

so that
$$\{(R+\delta)^2 - r^2\}^{\frac{1}{2}} = (R+\delta)\operatorname{sn} a,$$

and
$$\operatorname{dn}^2 a = 1 - \frac{4R\delta}{(R+\delta)^2 - r^2} \cdot \frac{(R+\delta)^2 - r^2}{(R+\delta)^2} = \left(\frac{R-\delta}{R+\delta}\right)^2,$$

or
$$(R+\delta)\operatorname{dn} a = R - \delta \quad \text{and} \quad (R+\delta)(1+\operatorname{dn} a) = 2R.$$

From the first figure we have

$$PT + TQ = 2R\sin(\phi - \theta).$$

Hence substituting from the foregoing equations

$$(R+\delta)\operatorname{sn} a\,(\operatorname{dn} u + \operatorname{dn} v) = (R+\delta)(1+\operatorname{dn} a)(\operatorname{sn} v\ \operatorname{cn} u - \operatorname{cn} v\ \operatorname{sn} u),$$

or
$$\frac{\operatorname{sn}(u+a)\operatorname{cn} u - \operatorname{cn}(u+a)\operatorname{sn} u}{\operatorname{dn}(u+a) + \operatorname{dn} u} = \frac{\operatorname{sn} a}{1+\operatorname{dn} a},$$

which is the addition formula (39) where $u_1 = u + a$ and $u_2 = -u$.

If we attempt to inscribe a polygon in the outer circle whose sides shall touch the inner circle, beginning at P whose argument is u, the successive angular points will have arguments $u+a$, $u+2a$, $u+3a$, and so on.

If P_n coincides with P, $u + na = u + 2K$, because θ has been increased by π, and therefore u by $2K$. In particular if the polygon is to have four sides, we have $4a = 2K$ or $a = \frac{K}{2}$.

We have also $\operatorname{sn} K = 1$, $\operatorname{cn} K = 0$, $\operatorname{dn} K = k'$.

Hence from equation (46),

$$\operatorname{sn}\tfrac{1}{2}K = \left[\frac{1}{1+k'}\right]^{\frac{1}{2}} \quad \text{and} \quad \therefore\ \operatorname{cn}\tfrac{1}{2}K = \left(\frac{k'}{1+k'}\right)^{\frac{1}{2}}.$$

Also from equation (51),

$$k'^2 = 1 - k^2 = \{(R-\delta)^2 - r^2\}/\{(R+\delta)^2 + r^2\},$$

and

$$\operatorname{cn} a = r/(R+\delta).$$

Hence

$$\frac{k'}{1+k'} = \frac{r^2}{(R+\delta)^2};$$

$$\therefore\ k' = \frac{r^2}{(R+\delta)^2 - r^2};$$

$$\therefore\ \frac{(R-\delta)^2 - r^2}{(R+\delta)^2 + r^2} = \left[\frac{r^2}{(R+\delta)^2 - r^2}\right]^2;$$

$$\therefore\ r^4 = \{(R-\delta)^2 - r^2\}\{(R+\delta)^2 - r^2\}$$

$$= (R^2 + \delta^2 - r^2)^2 - 4R^2\delta^2;$$

$$\therefore\ 4R^2\delta^2 = (R^2 + \delta^2 - r^2)^2 - r^4$$

$$= (R^2 + \delta^2)(R^2 + \delta^2 - 2r^2),$$

or

$$2r^2(R^2 + \delta^2) = (R^2 - \delta^2)^2.$$

When this condition is satisfied, since u is not fixed, we can begin at any point on the outer circle and describe a quadrilateral whose sides shall touch the inner circle.

The equations (48), (49), (50) can also be illustrated geometrically by making the inner circle a point.

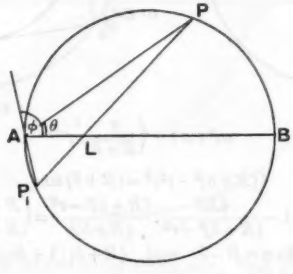

FIG. 4.

If $r=0$, we have $\operatorname{cn} a = 0$.

Hence $a = K$, and if L be the point on AB and PL be produced to meet the circle again in P_1, the arguments of P and P_1 are u and $u+K$.

Also

$$\frac{AL}{LB} = \frac{R-\delta}{R+\delta} = \operatorname{dn} a = \operatorname{dn} K = k'.$$

We have

$$\operatorname{sn} u = \sin\theta = BP/BA,$$

$$\operatorname{cn} u = \cos\theta = AP/BA,$$

$$\operatorname{dn} u = LP/LB \quad \text{from equation (53).}$$

Also travelling along the arc PAP_1,

$$\operatorname{sn}(u+K) = BP_1/BA,$$

$$\operatorname{cn}(u+K) = -AP_1/BA\ ; \quad \because\ \phi \text{ is obtuse};$$

$$\operatorname{dn}(u+K) = LP_1/LB, \quad \text{from equation (54).}$$

From the similar triangles BP_1L, APL,

$$BP_1/BL = PA/PL\ ;$$

$$\therefore\ \operatorname{sn}(u+K) = \frac{BL}{BA}\cdot\frac{PA}{LP} = \frac{AP}{BA}\cdot\frac{LP}{BL} = \frac{c}{a}.$$

Again, $$PL \,.\, LP_1 = BL \,.\, LA\,;$$

$$\therefore\ \mathrm{dn}\,(u+K) = \frac{LP_1}{LB} = \frac{BL\,.\,LA}{PL\,.\,LB} = \frac{k'}{d}.$$

Also $$\frac{AP_1}{AL} = \frac{BP}{PL},$$ from the similar triangles AP_1L, LBP.

Hence $$\mathrm{cn}\,(u+K) = -\frac{AL}{LB}\cdot\frac{BP}{BA} \div \frac{LP}{LB} = -\frac{k's}{d}.$$

12. *The pendulum.*

If the bob (of mass m) is released from rest when the string (of length l) makes an angle α with the vertical, the equation of energy gives

$$\tfrac{1}{2}m\,(l\dot\theta)^2 = mgl\,(\cos\theta - \cos\alpha)$$

where θ is a subsequent angle of inclination;

$$\therefore\ \dot\theta^2 = \frac{4g}{l}\,(\sin^2\tfrac{1}{2}\alpha - \sin^2\tfrac{1}{2}\theta);$$

let $$\sin\tfrac{1}{2}\theta = x\sin\tfrac{1}{2}\alpha.$$

then $$\tfrac{1}{2}\dot\theta\cos\tfrac{1}{2}\theta = \dot x\sin\tfrac{1}{2}\alpha,$$

whence, on substitution,

$$\dot x^2 = \frac{g}{l}(1-x^2)(1-\sin^2\tfrac{1}{2}\alpha\, x^2).$$

The solution of this is

$$x = \mathrm{sn}\left(t\sqrt{\frac{g}{l}}\right), \text{ where } k = \sin\tfrac{1}{2}\alpha,$$

while the period of a complete oscillation, $4K$, is

$$2\pi\sqrt{\frac{l}{g}}\left[1 + \left(\frac{1}{2}\right)^2\sin^2\frac{1}{2}\alpha + \left(\frac{1\,.\,3}{2\,.\,4}\right)^2\sin^4\frac{1}{2}\alpha + \ldots\right].$$

13. *The analytical geometry of the ellipse,* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

If $k = e$, the point $(a\,\mathrm{sn}\,u,\ b\,\mathrm{cn}\,u)$ will always lie on the curve, and the equation of the tangent at u is

$$\frac{x}{a}\mathrm{sn}\,u + \frac{y}{b}\mathrm{cn}\,u = 1. \tag{56}$$

To find the coordinates of the points of intersection T of the tangents at u and v, say, it is more convenient to put $u = \alpha+\beta$, $v = \alpha-\beta$. The tangents at u and v are then

$$\frac{x}{a}\mathrm{sn}\,(\alpha\pm\beta) + \frac{y}{b}\mathrm{cn}\,(\alpha\pm\beta) = 1.$$

Solving by addition and subtraction we have, using equations (43) and (45),

$$\frac{x}{a}s_1c_2d_2 + \frac{y}{b}c_1c_2 = 1 - k^2s_1^2s_2^2,$$

and $$\frac{x}{a}s_2c_1d_1 = \frac{y}{b}s_1s_2d_1d_2;$$

$$\therefore\ \frac{x/a}{s_1s_2d_2} = \frac{y/b}{s_2c_1} = \frac{1-k^2s_1^2s_2^2}{s_1^2s_2c_2d_2^2 + s_2c_1^2c_2} = \frac{1-k^2s_1^2s_2^2}{s_2c_2\{s_1^2(1-k^2s_2^2) + 1 - s_1^2\}};$$

$$\therefore\ \frac{x}{a} = s_1\frac{d_2}{c_2} \quad\text{and}\quad \frac{y}{b} = \frac{c_1}{c_2}. \tag{57}$$

Eliminating α between these equations we get

$$\frac{x^2}{a^2\left(\dfrac{d_2}{c_2}\right)^2} + \frac{y^2}{b^2\left(\dfrac{1}{c_2^2}\right)} = 1.$$

But $a^2\left(\frac{d_2}{c_2}\right)^2 - b^2\left(\frac{1}{c_2}\right)^2 = \frac{a^2(1-e^2s_2^2)-b^2}{c_2^2} = \frac{(a^2-b^2)(1-s_2^2)}{c_2^2} = a^2-b^2.$ (58)

Hence this system of conics is confocal with the ellipse.

Eliminating β between the equations we get, from equation (58),

$$\frac{x^2}{s_1^2} - \frac{y^2}{c_1^2} = a^2 - b^2,$$

which is also a confocal system.

If, then, $u \pm v$ is constant, the intersection of the tangents at the points u and v traces a confocal conic.

Equation (56) may be written

$$\frac{a\,\mathrm{sn}\,u - x}{a\,\mathrm{cn}\,u} = \frac{b\,\mathrm{cn}\,u - y}{-b\,\mathrm{sn}\,u},$$

and each of these fractions is equal to

$$\frac{\sqrt{(a\,\mathrm{sn}\,u-x)^2+(b\,\mathrm{cn}\,u-y)^2}}{\sqrt{a^2(1-s^2)+a^2(1-e^2)s^2}} = \frac{r}{a\,\mathrm{dn}\,u},$$

where r is the distance from the point of contact of any point (x, y) on the tangent.

If, then, the distance TP is called t_1, we may put

$$r = t_1, \quad \frac{y}{b} = \frac{c_1}{c_2}, \quad \text{and} \quad u = \alpha + \beta$$

in the equation

$$\frac{b\,\mathrm{cn}\,u - y}{-b\,\mathrm{sn}\,u} = \frac{r}{a\,\mathrm{dn}\,u}.$$

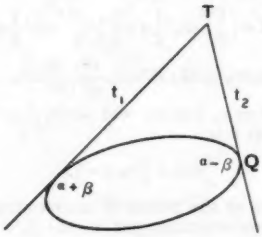

FIG. 5.

We have

$$\begin{aligned}\frac{y}{b} - \mathrm{cn}\,u &= \frac{c_1}{c_2} - \frac{c_1c_2 - s_1s_2d_1d_2}{1-k^2s_1^2s_2^2} \\ &= \frac{c_1^2 - c_1s_2^2(1-d_1^2) - c_1c_2^2 + s_1s_2c_2d_1d_2}{1-k^2s_1^2} \\ &= \frac{s_2d_1}{c_2}\cdot\frac{s_1c_2d_2 + s_2c_1d_1}{1-k^2s_1^2s_2^2} \\ &= \frac{s_2d_1}{c_2}\,\mathrm{sn}(\alpha+\beta) = \frac{s_2d_1}{c_2}\,\mathrm{sn}\,u.\end{aligned}$$

Hence

$$t_1 = a\frac{s_2d_1}{c_2}\,\mathrm{dn}(\alpha+\beta). \tag{59}$$

Similarly

$$t_2 = a\frac{s_2d_1}{c_2}\,\mathrm{dn}(\alpha-\beta). \tag{60}$$

These results are much simpler than those obtained by the use of the eccentric angle which involve heavy surds. In fact, the advantage of using

elliptic functions is that it enables one to rationalise two quadratic surds at once, the rationalisation being really contained in the second and third equations of definition (26) and (27).

From equations (59) and (60) we have

$$t_1+t_2=2as_2d_1^2d_2/c_2(1-k^2s_1^2s_2^2), \tag{61}$$

$$t_1-t_2=-2ae^2s_2^2s_1c_1d_1/(1-k^2s_1^2s_2^2). \tag{62}$$

14. *The arc of the ellipse.*

In equation (2) put $x=\operatorname{sn}u$, where $k=e$.

Then
$$\frac{dx}{d\sigma}=\frac{dx}{du}\cdot\frac{du}{d\sigma};$$

$$\therefore\ \operatorname{cn}u\ \operatorname{dn}u\ \frac{du}{d\sigma}=\frac{\operatorname{cn}u}{\operatorname{dn}u},$$

or
$$\frac{d\sigma}{du}=\operatorname{dn}^2u.$$

This equation cannot be integrated in terms of the preceding functions, and so forms the definition of a new function called Eu.

In fact
$$\frac{dEu}{du}=\operatorname{dn}^2u \quad \text{and} \quad E(0)=0.$$

Hence for the ellipse the length of the arc measured from the end of the minor axis $(0, b)$, where $u=0$, is $a\,Eu$. The value of u at the extremity of the major axis $(a, 0)$ is such that $\operatorname{sn}u=1$ and is therefore K. This value of Eu is called E, and from the equations of definition

$$E=\int_0^K \operatorname{dn}^2u\,du$$

while the whole circumference of the ellipse is $4aE$.

If we put $x=\sin\theta$ in equation (2) we get, on integrating,

$$\sigma=\int_0^\theta \sqrt{1-e^2\sin^2\theta}\,d\theta,$$

so that
$$E=\int_0^{\frac{\pi}{2}} \sqrt{1-e^2\sin^2\theta}\,d\theta,$$

and this can be expanded as an infinite series just as K was.

The function Eu has an addition formula.

Let $u_1+u_2=b$, a constant. Then, by definition,

$$\begin{aligned}\frac{d}{du_1}(Eu_1+Eu_2)&=d_1^2-d_2^2\\&=-k^2(s_1^2-s_2^2)\\&=-k^2\operatorname{sn}(u_1+u_2)(s_1c_2d_2-s_2c_1d_1)\ \text{by equation (41)}\\&=k^2\operatorname{sn}b\,\frac{d}{du_1}(s_1s_2).\end{aligned}$$

Hence $Eu_1+Eu_2-k^2s_1s_2\operatorname{sn}b$ is constant, and putting $u_1=b$, $u_2=0$ its value is Eb.

Hence
$$Eu_1+Eu_2-E(u_1+u_2)=k^2\operatorname{sn}u_1\operatorname{sn}u_2\operatorname{sn}(u_1+u_2)$$

and
$$Eu_1-Eu_2-E(u_1-u_2)=-k^2\operatorname{sn}u_1\operatorname{sn}u_2\operatorname{sn}(u_1-u_2).$$

Adding and subtracting, we have

$$2Eu_1-E(u_1+u_2)-E(u_1-u_2)=2k^2s_1s_2^2c_1d_1/(1-k^2s_1^2s_2^2)$$

and
$$2Eu_2-E(u_1+u_2)+E(u_1-u_2)=2k^2s_1^2s_2c_2d_2/(1-k^2s_1^2s_2^2).$$

Putting $u_1=\alpha$ and $u_2=\beta$, and changing signs throughout, we have

$$E(\alpha+\beta)+E(\alpha-\beta)-2E\alpha=-2k^2s_1s_2^2c_1d_1/(1-k^2s_1^2s_2^2)$$
$$=(t_1-t_2)/a \text{ from equation (62)}; \quad (63)$$
$$E(\alpha+\beta)-E(\alpha-\beta)-2E\beta=-2k^2s_1^2s_2c_2d_2/(1-k^2s_1^2s_2^2);$$

and this is easily verified as being equal to

$$(t_1+t_2)/a-\frac{2\,\mathrm{sn}\,\beta\,\mathrm{dn}\,\beta}{\mathrm{cn}\,\beta}. \quad (64)$$

If then $\alpha+\beta$, $\alpha-\beta$ are the arguments of the two points P and Q, the tangents at which meet in T, and if B is the point from which the arcs are being measured, we have, from equation (64), when T traces a confocal ellipse, so that β is a real constant,

$$\text{arc } BP-\text{arc } BQ-2aE\beta=TP+TQ-2a\,\mathrm{sn}\,\beta\,\mathrm{dn}\,\beta/\mathrm{cn}\,\beta,$$

or
$$TP+TQ-\text{arc } PQ=\text{a constant}.$$

And when T traces a confocal hyperbola so that a is a real constant, we have, from equation (63),

$$\text{arc } BP+\text{arc } BQ-TP+TQ=\text{a constant}$$
$$=\text{twice arc } BR$$

if R is the point of intersection of the hyperbola and ellipse between P and Q. Thus

$$TP-\text{arc } RP=TQ-\text{arc } RQ.$$

I have to acknowledge indebtedness to Prof. A. C. Dixon's *Elliptic Functions* for some of the above work, and to my colleague Mr. D. E. Collier, M.A., for some valuable suggestions.

N. M. GIBBINS.

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687. Adrian Metius, born Dec. 9th, 1571, describes his father as Adriaan Anthonisz, *i.e.* Adrian the son of Anthony. Lalande gives the father's name as Adrian, but Montucla hands it down as Peter (*Penny Cyclopaedia*, xv. p. 143). De Morgan (*Budget of Paradoxes*, 1872, p. 62) tells us that Peter was the first to find that π lies between $\frac{333}{106}$ and $\frac{377}{120}$, and that he "took the liberty of taking the mean of both numerators and denominators, giving $\frac{355}{113}$." But $\frac{355}{113}$ is said to have been discovered by Tsu Ch'ung-chih (428-499). Moreover, it seems that Montucla (*Recherches sur la Quadrature du cercle*, 1754, p. 42) invented Peter. It came about thus. Adrian, in his *Arithmetica et Geometria practica* (Leyden), 1640, refers to his father's "discovery" in the words: *Parens P.M. Illustrium DD. Ordinum Confoederatarum Belgiae Provinciarum Geometra . . .*, which is, being interpreted: My Father, of pious memory (*piae memoriae*), Geometer to the illustrious lords of the confederate provinces of the Pays-Bas. . . . A hundred and fifty years later, Bierens de Haan (cf. *Mathesis*, 1909, p. 17) pointed out how P(etrus) M(etius) came into existence, as the pseudonym of Anthony Metius.

688. *Solution of problem of Duration of Play.*

"Laplace says that if $2m$ is less than n *it is clear* that the fraction

$$[(1+\alpha)^{n-2m}-(1-\alpha)^{n-2m}]/[(1+\alpha)^n-(1-\alpha)^n]$$

diminishes as α increases. . . .

Laplace treats the same question in the *Théorie Analytique des Probabilités*, page 406; there also the difficulty is dismissed with the words *on verra facilement.* In the memoir prefixed to the fourth volume of Bowditch's *Translation of the Mécanique Céleste*, page 62, we read:

Dr. Bowditch himself was accustomed to remark, 'Whenever I meet in La Place with the words "Thus *it plainly appears*" I am sure that hours, and perhaps days of hard study will alone enable me to discover *how* it plainly appears.'"—Todhunter, *History of the Theory of Probability*, pp. 476-8 [per Miss E. J. Ternouth].

HISTORY OF THE THEORY OF IMAGINARY AND COMPLEX QUANTITIES.

By G. Windred.

It is worthy of note that the imaginary was introduced into mathematics in the course of strictly non-utilitarian researches on the theory of quadratic equations, being for a long time rejected by the mathematicians as an entirely useless conception. This formal condemnation of the imaginary seems less extraordinary when we recollect that at the time of its inception, and indeed for long afterwards, serious scruples were experienced by some mathematicians even regarding the admission of negative quantities in their results. The last writers who objected to the use of negative quantities were Baron Maseres, a Fellow of Clare College, Cambridge, and William Frend, a second wrangler. Both these men had written extensively in condemnation of the use of negative quantities; their attitude in this matter is clearly evinced by the following extract from the preface to Frend's *Principles of Algebra*, published in 1796, at which period, strangely enough, the notion of the imaginary was firmly established.

"You may put a mark before one, which it will obey; it submits to be taken away from a number greater than itself, but to attempt to take it away from a number less than itself is ridiculous. Yet this is attempted by algebraists, who talk of a number less than nothing; of multiplying a negative number into a negative number and thus producing a positive number; of a number being imaginary. Hence they talk of two roots to every equation of the second order, and the learner is to try which will succeed in a given equation: they talk of solving an equation, which requires two impossible roots to make it soluble; they can find out some impossible numbers, which, being multiplied together, produce unity. This is all jargon, at which common sense recoils; but, from its having been once adopted, like many other figments, it finds the most strenuous supporters among those who love to take things upon trust, and hate the colour of a serious thought."

We may remark that in this discourse Frend overlooks the fact that the science of pure number is essentially abstract; if this were not so we should be obliged to reject all powers of numbers higher than the third in dealing with linear magnitudes on account of the fact that we exist in a world of three dimensions, and a four-dimensional object is therefore not amenable to the presentational continuum normal to us. Frend's objections are further outlined in his *True Theory of Equations*, 1799, while those of Maseres recur frequently in his extensive writings. Maseres is a very outstanding personality in the history of mathematics. Born in 1731, he was a contemporary of William Frend (1757-1841), whose opinions concerning the negative quantity he shared. Maseres was the fourth wrangler of 1752, first Chancellor's medallist in classics, and was appointed Cursitor Baron of the Exchequer in 1773. In 1758 he published *A Dissertation on the use of the Negative Sign in Algebra*, setting forth in great detail his objections to this symbol. This publication was followed by the *Scriptores Logarithmici*, published in six volumes, London, 1791-1807, which comprises a collection of numerous tracts on the theory of logarithms, and also some writings on the binomial theorem. In the year 1823, a year before his death, a further collection of optical papers, called the *Scriptores Optici* was published for Maseres under the editorship of Charles Babbage, afterwards Lucasian Professor of Mathematics at Cambridge. In his dissertation, Maseres remarks that "A single quantity can never be marked with either of those signs, or considered as either affirmative or negative; for if any single quantity, as b, is marked either with the sign + or with the sign − without assigning some other quantity, as a, to which it is to be added, or from which it is to be subtracted, the mark will have no meaning or signification." Anent

the views of Maseres and Frend,* George Peacock, who afterwards became Lowndean Professor of Astronomy at Cambridge University, was constrained to reply to the effect that the science of algebra may be considered as consisting of two parts, arithmetical and symbolical. In arithmetical algebra every combination of symbols must be reducible to an integer number, that is, any operation upon a number must result in a number, whereas in symbolical algebra the rules of arithmetical algebra are retained but the restrictions are removed; thus operations in symbolical algebra are possible for all values of the symbols used.

This statement of the problem clearly involves considerations of rational logic, if we are to perceive whether or not ascendance from the particular to the general is in this instance permissible.

The fact remains that these principles have become generally accepted, and in the course of our researches on the theory of equations it has been found necessary to accept the notions of both negative and imaginary quantities. This acceptance results not only in the logical completion of the arithmetical continuum but also in the completion of the theory of algebraical equations of the higher orders.

From this latter theory the science of the imaginary has sprung, and it is interesting to regard the rise of this science from the point of its first inception in the theory of equations.

The first written reference in the history of mathematics to the square root of a negative quantity occurs in the famous *Ars Magna*, published in 1545 by Hieronimo Cardano (1501-1576). This remarkable volume represents a great improvement upon all the previous mathematical publications dealing with the theory of equations, but it is questionable, if not even doubtful, whether certain parts of this work are original. The greater part of the work has been attributed to Nicolo of Brescia, afterwards called Tartaglia (1499-1557) who delayed the publication of his method for the solution of cubic equations in order that it might be included in a monumental work on algebra which he contemplated. The publication of Cardan's *Ars Magna*, containing Tartaglia's method, led to a bitter controversy. In the inevitable mathematical contest which resulted, Tartaglia was able clearly to demonstrate his mathematical superiority over Cardano and also his pupil Lodovico Ferrari (1522-1565), but Tartaglia's death prevented the completion of his proposed treatise.

The writings of Cardan are chiefly on the theory of quadratic and cubic equations, the solution of which led him to discuss complex roots. In the *Ars Magna* positive roots of an equation are termed *real*; negative roots are called *fictitious*; complex roots are called *sophistic*, and imaginary roots are entirely neglected. Cardan also proved that complex roots of an equation occur always in pairs, but beyond their application to the solution of equations he had no interest in these quantities and consequently did not develop their conception.

Similar researches to those of Cardan were made by Raphael Bombelli (*c.* 1530), who in 1572 published an Algebra containing a demonstration of the reality of the so-called imaginary quantity. He also assigned the value of the rational imaginary; and thus put the science of the imaginary quantity on a more exact and useful basis. His researches in this direction seem to represent the first attempt to put the theory of the imaginary into its proper place in analysis. The subsequent development of the theory owes much to these early fixations of ideas.

The next important writer on this subject was John Wallis (1616-1703). To his very original researches many important branches of mathematics are indebted. In 1649 he became Savilian Professor of Geometry at Oxford University, and was one of the original members of the Royal Society at its foundation in 1663. One of Wallis's greatest contributions to mathematical

* Also *v.* De Morgan's review of his father-in-law's book, *Budget of Paradoxes*, 1872, pp. 117-125.

science is his work on the theory of infinite series, in connection with which he introduced the term hypergeometric. His greatest published work, entitled *Arithmetica Infinitorum*, which appeared in 1655, had a great influence on Newton's investigation of the quadrature of curves. A perusal of this work led Newton to his discovery of the binomial theorem, by means of which the quadrature of curves is effected much more simply than by the method of interpolation hitherto employed. Wallis was considerably interested in the controversy between the followers of Newton and Leibniz regarding the authorship of the *Differential Calculus*, and in the preface to a volume of his works stated that the *calculus differentialis* of Leibniz was the method of fluxions produced by Newton.

In his *Algebra*, published at Oxford in 1685, and followed by a Latin edition in 1693, Wallis attempts a graphical representation of the complex numbers forming the roots of a real quadratic equation. These investigations were not successful, as he failed to discover a consistent and general graphical construction for all complex values.*

Accurate graphical methods for the representation of imaginaries were not evolved until more than a century after Wallis's researches.

The most important contributor to the theory next following Wallis was the great Leonhard Euler (1707-1783). At the University of Basel he came under the charge of Johann Bernoulli, with whom he afterwards held extensive correspondence on the subject of imaginary logarithms. Euler's researches on logarithms are very extensive. In his *Introductio in analysin infinitorum*, published in 1748, he abandoned the prevailing view of logarithms as terms of a corresponding series, and established them as exponents. This system was freely accepted, mainly in virtue of the influence and status of their inventor, although it is open to doubt whether an advance is made thereby. The Eulerian theory of logarithms undoubtedly involves serious analytical difficulties as compared with the former theory. At this time Leibniz considered that since a positive number larger than unity has a positive logarithm, and a positive number less than unity a negative logarithm, it is impossible to assign the logarithm of -1; thus he regarded $\log(-1)$ as an impossible logarithm. He further argued that if $\log(-1)$ is evaluable, then half $\log(-1)=\log\sqrt{-1}$, which he considered to be an absurdity. Bernoulli pointed out that since $dx : x :: -dx : -x$ there results by integration $\log(x)=\log(-x)$. It follows from this that the logarithmic curve $y=\log(x)$ has two symmetrical branches with reference to the y-axis. The correspondence on the subject of imaginary logarithms was continued with Bernoulli by Euler, who showed that the expression evolved by Bernoulli for the area of a circular sector becomes for a quadrant $\frac{2^2\log(-1)}{4\sqrt{-1}}$, which is evidently not in accordance with Bernoulli's estimate that $\log(-1)=0$. The correspondence between Bernoulli and Euler lasted from 1727 to 1731 and from the latter date Euler made extensive researches in the theory of imaginary quantities. In the year 1740 he was able to state, in a letter to Bernoulli, that

$$y=2\cos x=e^{x\sqrt{-1}}+e^{-x\sqrt{-1}},$$

and that these expressions are integrals of the differential equation

$$\frac{d^2y}{dx^2}+y=0.$$

This demonstration, together with the corresponding expressions for $\sin x$ was published in his *Miscellanea Berolinensia*, 1743, and in the famous *Introductio in analysin infinitorum*, 1748. In 1762 he published an article entitled *Sur les logarithmes des nombres négatifs et imaginaires* and in the year 1749 two

* The best account of Wallis's graphical constructions is contained in the article "Die geometrische Darstellung imaginärer Grössen bei Wallis," by G. Enestrom, *Bibliotheca mathematica*, series 3, vol. 7, 1906, pp. 263-269.

papers, *Recherches sur les racines imaginaires des équations* and *De la controverse entre Mons. Leibnitz et Bernoulli sur les logarithmes négatifs et imaginaires* were written by him. These last two papers were published in 1751 in the Berlin Memoirs; they do not seem to have carried any great conviction, or to have gone very far towards settling the various controversies. At this time D'Alembert became interested in the question, which suffered much aberration as a result of his highly involved arguments appealing severally to analytical, metaphysical, and geometrical concepts. D'Alembert did not however publish anything of a progressive nature in the theory of imaginaries; his attention was mainly occupied with the problems of applied mathematics, in which subject his fame is represented by the Principle bearing his name.

The famous Edward Waring (1734-1798) who was a senior wrangler, and afterwards Lucasian Professor of Mathematics at Cambridge, made numerous original contributions to algebraical theories, among the most notable of which is his Problem in the theory of numbers, which has received attention from a long succession of mathematicians including Fermat, Euler, Lagrange, and Jacobi down to the present day researches of Professor G. H. Hardy and others.* In his *Meditationes algebraicae*, published in 1770, Waring gives an original process for approximating to the values of imaginary roots. If in an equation x is represented approximately by $a+ib$, it may be replaced by

$$x=a+a^1+(b+b^1)i;$$

whereupon expanding and rejecting higher powers of a^1 and b^1 it is possible, by equating real and imaginary quantities, to obtain two equations yielding values of a^1 and b^1. Except for his enunciation of the relations which must of necessity exist between the coefficients of a quartic and quintic equation to yield two and four imaginary roots, Waring made no further contributions to the theory of the imaginary, being undoubtedly more greatly interested in the development and analysis of the equations which had brought the subject of imaginaries to his attention.

A very important, though perhaps not very ingenuous work entitled *Meditationes de quantitatibus imaginariis construendis et radicibus imaginariis exhibendis* was published in 1753 by Heinrich Kühn (1690-1769). This work, while displaying several interesting propositions, is not of high mathematical standard; it contains much of a doubtful nature that might, in the writer's interests as a mathematician, have been omitted. In certain parts it is also not original; it leaves the impression that Kühn's method of reasoning had been considerably influenced by the writings of John Wallis sixty-eight years previously. An interesting assumption is made by Kühn with reference to four squares contained in rectangular co-ordinate axes in such wise that two sides of each square correspond to a co-ordinate. According to Kühn, the areas of the respective squares are $a\times a$, $-a\times(+a)$, $(-a)\times(-a)$, $+a\times(-a)$, so that if the area of a square is $-a^2$, it follows that the *latus* (side) or *radix* (root) is properly expressed by $\sqrt{-a^2}$. Anent this demonstration Kühn remarks: "*Hic satis speciose objici potest ejusmodi rationis radices* $x=\mp\sqrt{-a^2}$, *e.g.* $\mp\sqrt{-9}$ *esse mere imaginarias, impossibiles et inassignabiles propterea quod ex* $-a^2$ *nullo modo radix quadrata extrahi possit, nec enim eam esse* $-a$ *nec* $+a$ *cum* $-a\,.\,-a$ *item* $+a\,.\,+a$ *det quadratum positivum* $=a^2$ *atque adeo omnia quadrata realia aut assignabilia esse positiva.*" Kühn also gives a very detailed discussion of the affected quadratic, in the course of which he establishes an interesting proposition on the lines of his former argument. The sum or difference between two positive or negative squares may be represented by another square, as also may be half the sum or half the difference of the original squares. For the solution of the equation

$$x^2+bx+c=0$$

* *Vide* G. H. Hardy, *Some Famous Problems of the Theory of Numbers*, Oxford, 1920, p. 14 *et seqq.*

the problem therefore resolves into constructing positive or negative squares, x_1^2, x_2^2, such that

$$\tfrac{1}{2}(x_1^2+x_2^2)=\tfrac{1}{2}b^2-c,\ \tfrac{1}{2}(x_1^2-x_2^2)=b\sqrt{\tfrac{1}{4}b^2-c}.$$

From this it follows that in the case where $b^2>4c$ the two squares are positive, and their *radices*, which represent the required roots of the equation are determined. In the case where $b^2<4c$ the construction is impossible, but a positive square equivalent to $\frac{1}{4}b^2$ and a negative square equivalent to $\frac{1}{4}b^2-c$ may be found; so that the roots will still be determinate, though not now in the form of sides to an actual square. From these examples of Kühn's work it will be appreciated that no great advance in the science is represented thereby. His results are substantially those obtained by Wallis in his *Algebra* of 1685.

It remained for Caspar Wessel (1745-1818) to make the next important contribution to the theory. This investigator was by profession a surveyor in the employ of the Danish Academy of Science, and was also well read in Roman law. It was not until 1797, when he was fifty-two years old, that he made his single contribution to mathematics in the form of a paper, *Om Directionens analytiske Betegning*, presented to the Royal Danish Academy and published in their *Transactions* in 1799. After a lapse of ninety-eight years, during which the memoir attracted little or no attention, it was rediscovered by an antiquary, and two years later, on the centenary of its birth, a French translation was issued by the Danish Academy. The principal object of Wessel's memoir is to evolve a calculus of vectors. This object is substantially realised, and the results obtained are among the most valuable hitherto. The memoir contains the earliest published graphical representation of the imaginary $\sqrt{-1}$ and the general complex number of the form $a+b\sqrt{-1}$. Wessel employed the conception of unit vectors on the co-ordinate axes and assumed that the generation of a given vector from the unit vector was of necessity to be regarded as a combined rotation and alteration in the length of the unit vector. In certain respects, especially in connection with vectors in three dimensions, the results of Wessel seem to foreshadow the elements of Hamilton's quaternions. In developing his thesis Wessel denotes the unit vectors by 1, ϵ, -1, and $-\epsilon$ respectively; thus he finds that $-1=\epsilon^2$, and hence replaces ϵ by $\sqrt{-1}$. Proceeding with this analogy he is then able to establish the fundamental operations of algebra with complex forms, thus:

$$(a+b\epsilon)(c+d\epsilon)=(ac-bd)+(ad+bc)\epsilon,$$

$$\frac{a+b\epsilon}{c+d\epsilon}=\frac{ac+bd}{a^2+d^2}+\frac{bc-ad}{c^2+d^2}\epsilon,$$

$$(\cos v+\sin v\epsilon)^{\frac{n}{m}}=\cos\frac{n}{m}v+\sin\frac{n}{m}v\epsilon.$$

In addition to these fundamental results the memoir contains interesting analysis of simple functions of complex numbers. The chief value of the memoir lies, however, in its accurate fundamental definitions and principles.

We come next upon the famous Augustin Louis Cauchy (1789-1857), one of the most prolific mathematical writers who ever lived. His researches in practically every part of both pure and applied mathematics inevitably embraced the theory of imaginaries. He inquired into the question of the circumstances governing the application of complex numbers; the determination of the number of real and imaginary roots of an equation; and the method of establishing a connection between different branches of a subject by assigning complex values to independent variables. Cauchy was the first writer to give a rigorous proof of Taylor's theorem. His most valuable contributions are to the theory of functions and the theory of equations.

A further important and much neglected memoir entitled *Essai sur une manière de représenter les quantités imaginaires dans les constructions géo-*

métriques was published in 1806 by Jean Robert Argand (1768-1822). This memoir is somewhat similar in certain parts to that of Wessel, but is inferior in the matter of mathematical rigor. Argand commences by considering the imaginary in its directive relation to positive and negative quantities, and shows its correspondence to vertical direction when regarded in this manner. He then considers the rules of addition and multiplication applying to complex or directed quantities, therein proceeding in a similar manner to that adopted by Wessel. The memoir also contains a number of trigonometrical identities which are, however, of lesser importance on account of their indifferent development.

The fact that the validity of solutions is in no way affected by the introduction of imaginaries among ordinary quantities was first discovered by the geometer Gaspard Monge (1746-1818), whose fame rests in the importance of his contributions on the theory of surfaces and curves of double curvature.

In the same year as the publication of Argand's *Essai* there appeared in the *Philosophical Transactions* of the Royal Society a most extraordinary paper entitled "*Mémoire sur les quantités imaginaires*," presented by William Morgan on behalf of the Abbé Buée. It must be stated that this paper serves to indicate little more than the fact that the Abbé's ideas concerning imaginaries are decidedly unorthodox. This fact is amply attested by the following passage: "*Si, par exemple, j'exprime un temps futur par t, et un temps passé par* $-t$, $t\sqrt{-1}$ *ne peut rien signifier, puisque le présent, qui est la qualité moyenne entre le futur et le passé, n'est qu'un instant indivisible et qui n'a d'autre expression que* 0. . . ." The Abbé then concludes that present time is composed of $t \times \frac{1}{2}\sqrt{-1}$ and $-t \times \frac{1}{2}\sqrt{-1}$, where t and $-t$ have the significance of future and past time respectively. This contribution evidently cannot be admitted as at all valuable, at least from the mathematical point of view.

In the year 1809 appeared the fifth edition of the *Rudiments of Mathematics*, by William Ludlam (1718-1788), a Fellow of St. John's College, Cambridge. The prefatory notes to this interesting work reflect very clearly the uncertainty which existed even at this time concerning the use of the negative sign. Anent this symbol, and referring to Baron Masere's *Dissertation on the Negative Sign*, Ludlam says:

"It is, however, the subject of a quarto volume, published by an excellent Mathematician; one who has explained some of the abstrusest parts of Algebra on the clearest principles, and who is above valuing himself on a dexterity in the manœuvre of a parcel of symbols, such that no man can form an idea of what they represent." The last dictum might readily apply to the Abbé Buée, but to very few of the other investigators. The next notable reference to imaginaries occurs in the *Traité des Propriétés projectives des figures*, compiled by Jean Victor Poncelet (1788-1867) while a prisoner of war at Saratoff, and published at Paris in 1822, eight years after his return to France. This writer concerned himself almost entirely with the geometrical aspect of the imaginary in its application to the geometry of the complex domain. Other famous contributors to this subject, which lies outside the scope of the present paper, were Gregory and Walton in England; Chasles, Marie, Darboux and numerous other writers in France; Von Staudt and Study in Germany.*

The year 1828 was productive of two notable contributions to the theory. In that year the Rev. John Warren published at Cambridge *A Treatise on the Geometrical Representation of the Square Roots of Negative Quantities*. The work contains a thorough and carefully written examination of important parts of the theory, to the literature of which it forms an important addition. Warren uses the word "quantity" expressly as representing a vector, and, like the majority of contemporary investigators, considers the whole of the vector, from the origin to the terminal point, not the latter point (x, y) only.

* For detailed references to these investigators, cf. J. L. Coolidge, *Geometry of the Complex Domain*, Oxford, 1924, p. 32 *et seqq.*

In addition to graphical constructions he also pays some attention to points in the algebraical theory which had hitherto been neglected or indifferently developed.

Also in 1828 was published in Paris a work by C. V. Mourey entitled *La vraie théorie des quantités négatives et des quantités prétendues imaginaires.* In this work the theory of vectors is systematically studied. Mourey seems to be the first writer to appreciate the necessity of specifying the conditions of equality between vectors. He points out that for equality between two directed quantities, or vectors, it is necessary that equality exists both in length and direction. The forms of notation employed by this writer have not survived, but several of the problems he discusses are of distinct mathematical value. He defines the expression $\left(\frac{x}{y}\right)_{\frac{m}{n}}$ as an operator which acts on a vector in such wise as to produce an alteration of length in the ratio $x : y$, and a rotation through the fraction $\frac{m}{n}$ of a right angle. In this respect his notation is analogous, even if not similar, to the modern one.

On April 15th, 1831, the great Karl Friedrich Gauss (1777-1855) presented to the Royal Society of Göttingen a memoir entitled *Theoria residuorum biquadraticorum, commentatio secunda.* The question of the significance of $\sqrt{-1}$ as an operator is herein discussed in its relation to co-ordinate systems by an ingenious demonstration based on the theory of transference between related series. The memoir also contains the first instance of the use of the form $a+bi$ to represent the general complex number, as well as a proof of the theorem that every equation has a root, either real or imaginary. In some respects it is an extension of the subject matter of his previous dissertation, *Demonstratio nova theorematis omnem functionem algebricam rationalem integram unius variabilis in factores reales primi vel secundi gradus resolvi posse,* published as a memoir in 1799 and again in his *Collected Works*, Göttingen, 1878, Vol. III. Gauss' researches on the theory of numbers, given in his *Disquisitiones Arithmeticae* of 1801, marked a new era in this subject. Certain parts of this work were rediscoveries of the previously published results of Lagrange and Euler. Gauss was eighteen years of age when he obtained by induction the famous law of quadratic reciprocity embracing the whole theory of quadratic residues. In all, he gave six different proofs of this theorem, which had been imperfectly stated previously by Euler. In 1807 Gauss became professor of Astronomy in the University of Göttingen. He first introduced the term "complex numbers" and the letter i to represent $\sqrt{-1}$. His scientific researches are as important as those in pure mathematics. At the Göttingen Observatory he carried out extensive researches on terrestrial magnetism. In applied mathematics he gave considerable attention to the theory of attraction, in which subject exist several theorems bearing his name.

We come at last to the contribution of the great genius, Sir William Rowan Hamilton (1805-1865), published in the *Transactions of the Royal Irish Academy*, Vol. XVII., 1837, pt. 2, and entitled "Theory of Conjugate Functions, or Algebraic Couples; with Preliminary Essay on Algebra as the Science of Pure Time." In view of the unique and highly important nature of this memoir, notwithstanding the fact of its complete neglect at the hands of subsequent writers, it will be pertinent here to recount the broad outline of the thesis.

All physical experiences of an individual are accompanied by an intuition of Time, which forms a necessary component of the elements required fully to specify any physical occurrence or presence. The complete specification of any physical entity involves not only an answer to the locative question, but also to the question of time of occurrence; or in the case of a pre-existing entity, to the time of commencement of observation, and perhaps also to the time of duration of observation. Such conceptions are purely abstract in character

and must be abstracted from the space time intuitions of the individual. If the temporal intuition corresponds to the present time of the percipient then he regards his observation as an act of attention. Incidents preceding the present time intuition of the percipient are regarded by him as belonging to the past. Incidents succeeding to his present time intuition are purely anticipatory, and are consequently not amenable to spatial or temporal specification. Thus in the temporal intuition of an individual there exists a qualitative distinction between past, present and future, between which he is able to distinguish by abstracting from his own private intuitions of time. It is manifest that for the purposes of logical abstraction the temporal reference continuum must be idealised. This idealisation can only be obtained by an appeal to the private temporal intuition of the individual percipient. Thus in the abstract scheme of inquiry we are led to express times in their order of occurrence, so that Time is represented as an independent variable. The values of this variable are, from the limitless nature of past and future, evidently comparable to the arithmetical number continuum ; and thus any particular time may be represented in the abstract scheme by means of a pure number. In a similar manner an interval of time is expressible in the abstract scheme by means of the difference of two numbers, representing respectively the beginning and the end of the interval.

Hamilton commences by considering the possible relations of two pairs of such numbers *inter se*, and derives therefrom equations expressing analogies or non-analogies due respectively to equality or inequality between the moment pairs. This reasoning is extended to the case of combinations of two different analogies, or non-analogies, and thence to continued analogies, or equidistant series of moments. A system of abridged notation is developed whereby the difference $B-A$ between the moments B and A is represented by a, and likewise $D-C=b$. With this notation inverse relations are expressed through the medium of the symbol * Θ, which, prefixed to the a or b of a given ordinal relation, reverses the order of the relation. Thus, if $B-A=a$, then $A-B=\Theta a$, and if $D-C=b$, then $C-D=\Theta b$. By an extension of this reasoning, involving very lengthy and complex operations, which it would be quite impossible here even briefly to recapitulate, Hamilton develops a comprehensive algebra of intervals and ratios in the number-continuum corresponding to what he terms pure time ; that which we should now prefer to designate abstract time. The second part of the memoir is concerned with the development of the conception of moment-couples, which are defined as being of the form (A_1, A_2) where A_1 is a *primary moment* and A_2 a *secondary moment*, without regard as to whether A_2 follows, precedes or coincides with the primary moment. From this is developed the algebraical theory of the corresponding number-couples, including as a special case the couple $(-1, 0)$ which has two distinct square roots and no more namely:

$$(-1, 0)^{\frac{1}{2}}=(0, 1),$$
$$(-1, 0)^{\frac{1}{2}}=(0, -1).\dagger$$

These results are of considerable interest in relation to the modern theory of number couples, on account of the unique method of derivation. Hamilton denotes the principal square root of the couple (b_1, b_2) by the symbol $\sqrt{(b_1, b_2)}$, and thus obtains

$$\sqrt{(-1, 0)}=(0, 1)$$

which from the principle that every single number a is equivalent to a pure primary number-couple $(a, 0)$, may be expressed in the form

$$\sqrt{-1}=(0, 1).\ddagger$$

* Hamilton derived this symbol from the initial letter of the Latin word *Oppositio*, distinguished by a bar across it from the same letter used for other purposes.

† *Op. cit.* p. 417.

‡ *Op. cit.* p. 404.

Anent this demonstration Hamilton remarks "In the theory of single numbers, the symbol $\sqrt{-1}$ is absurd, and denotes an impossible extraction, or a merely imaginary number; but in the theory of couples, the same symbol $\sqrt{-1}$ is significant, and denotes a possible extraction, or a real couple, namely (as we have just now seen) the principal square root of the couple $(-1, 0)$." This is an excellent summing-up of an important fact which has escaped all but a comparatively few investigators of the subject. We might more generally express Hamilton's reasoning by remarking that any discussion of the validity or otherwise of a mathematical operation must relate to the appropriate mathematical domain. We have already considered the difficulties experienced by Maseres and Frend in connection with negative quantities. So far as their observations go, their inferences are in accordance therewith, but they make an unwarranted, and consequently illogical, extension of these inferences so as to include the whole conceptual number-continuum. Their mathematics is without reason confined by a postulation which limits the conception of number to a unidirectional, non-reversible operation. The removal of this erroneous postulate has rendered possible the logical completion of the pure number-continuum. In a similar manner, again considering the directive sense of number as in Hamilton's memoir, it is necessary to extend the meaning of number as applied to steps lying outside a straight line, and hence to adopt the Cartesian co-ordinate system for the representation of the new spatial continuum, which admits of the complete specification of any directed quantity.

It has been suggested that the underlying thought of Hamilton's conception of moment-couples was engendered by his study of Kant's *Critique* wherein it is stated that "Time and Space are two sources of knowledge from which various *a priori* synthetical cognitions can be derived. Of this, pure mathematics gives a splendid example in the case of our cognitions of space and its various relations. As they are both pure forms of sensuous intuition, they render synthetical propositions *a priori* possible." From this dictum Hamilton justly reasoned that since geometry is the science of space, then algebra is the science of time. The mathematical success of his memoir bears testimony to the accuracy of this reasoning.

It may be remarked that throughout Kant's Philosophy on the nature of arithmetical science the concept of the correspondence of pure mathematics to the intuition of time is very marked, although it is difficult, on account of his conflicting views of space and time, to evolve any complete principle therefrom. Anent the views of Kant on this subject Cassirer remarks that:

"Even discounting Kant's insistence upon the conceptual character of arithmetical science, and allowing that he derives arithmetical concepts and propositions from the *pure intuition of time*, this teaching, to whatever objections it may lie open, has certainly not the merely psychological meaning which the majority of its critics have ascribed to it...." "It is easily seen that Kant is only concerned with the 'transcendental' determination of the concepts of time, according to which it appears as the type of an ordered sequence. William Rowan Hamilton, who adopts Kant's doctrine, has defined algebra as 'science of pure time, or *order in progression*.'" *

Although we find no reference to Kant in Hamilton's memoir it is quite conceivable that Cassirer's estimate is a just one. The following statement of Kant is significant. "For if we represent time by a straight line extended to infinity, and simultaneous things at any point of time by lines successively erected perpendicular to the first line, the surface thus generated will represent the phenomenal world both as to substance and as to accidents." †

This is probably as near as we can get in other writings to the fundamental spirit of Hamilton's inquiry.

G. WINDRED.

* *Kant und die moderne Mathematik in Kant-Studien*, xii. (1907), p. 34. *Vide etiam*, N. K. Smith, *A Commentary to Kant's "Critique of Pure Reason"* (1918), p. 128 *et seqq.*

† *Ibidem*, p 135.

THE CONSTRUCTION FOR MEAN PROPORTIONAL.

By C. O. Tuckey, M.A.

One of the charms of elementary geometry is that there is no theorem so well known but that one may hope to discover improvements or at least interesting variations in the proof; and the same thing is true of the constructions.

In order to decide between variations in these it is necessary to have some way of measuring the complexity of a construction. This is done by counting the number of settings of the ruler or compass and the number of lines or arcs drawn.

Thus if we use S for setting and D for drawing,

to draw a circle centre C to pass through A needs $2S+D$ for each end of the compasses must be set;
to draw a circle centre C with radius equal to AB need $3S+D$;
to join AB needs $2S+D$;
to draw the perpendicular bisector of AB needs $S+D$ twice for the arcs (the radius being unspecified) and $2S+D$ for the bisector itself, making a total of $4S+3D$.

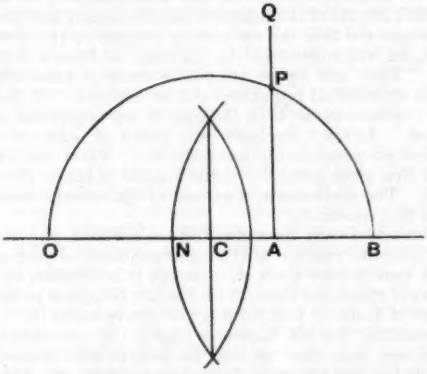

Fig. 1.

Now consider the construction of the mean proportional between OA and OB given in position on a line as in the figure.

The usual construction first finds C the mid-point of OB needing $4S+3D$; then draws the circle centre C, radius CO, needing $2S+D$.

After this $2S+D$ finds N, such that $AN=AB$, $2S+2D$ finds Q, $2S+D$ draws AQ and thus finds P.

This makes a total of $12S+8D$ to find P, or, if OP is to be drawn, $14S+9D$.

A shorter method finds the mean proportional between $\frac{1}{2}OA$ and $2OB$ by a construction which is essentially the same but much shorter, for it only needs $2S+D$ for the circle of centre B and $4S+3D$ for the perpendicular bisector of OA.

Thus R is found by $6S+4D$, or, if OR is to be drawn, $8S+5D$.

This construction, which has little more than half the complexity of the original method, is given in Lemoine's *Geometrographie*—a systematic account of economical methods for geometrical constructions, of whose somewhat elaborate notation the S, D notation used above is an attempted simplification.

The variation now to be described, which is the excuse for this note, is even simpler than the above method, for it requires only the drawing of *three circles all of the same radius.*

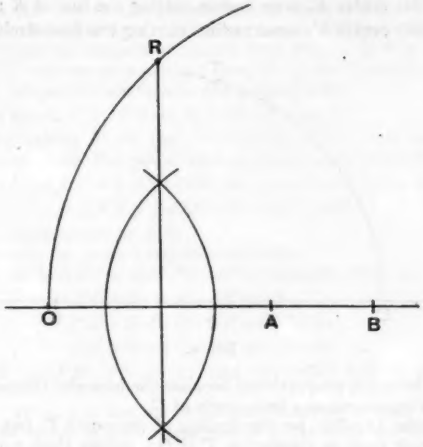

FIG. 2.

Draw the circles:

(i) centre B, radius BO;
(ii) centre A, same radius, cutting the line in J;
(iii) centre J, same radius.

Then the third circle cuts the first at the point R found above.

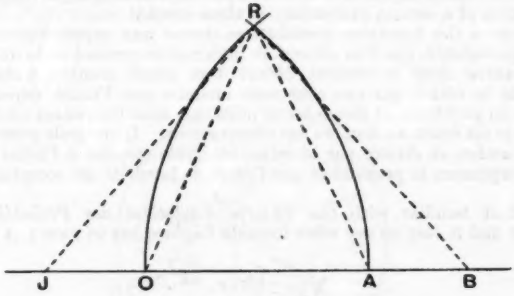

FIG. 3.

This construction needs $2S+D$ for the first circle and $S+D$ for each of the others, or in all $4S+3D$ to find R, or $6S+4D$ if OR is drawn.

If a construction of less complexity has been devised, I hope some reader of the *Mathematical Gazette* will supply it.

The proof that OR is the mean proportional between OA and OB is that the isosceles triangles OAR, ORB are similar, having a common base angle at O.

Another and, when it is possible, a nicer way of arranging the construction is to draw the three circles:

(i) centre A, radius AO;
(ii) centre B, same radius cutting the line at K;
(iii) centre K, same radius, cutting the first circle at T.

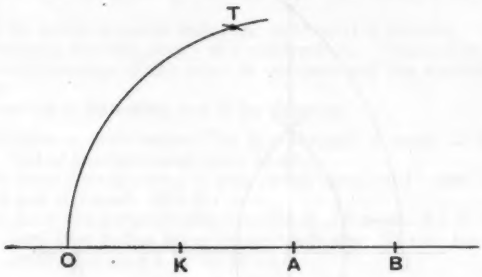

FIG. 4.

Then OT is the mean proportional because the isosceles triangles OAT, OTB are similar, having a common base angle at O.

This also needs $4S+3D$ for the finding of the point T, but it has the disadvantage that it fails to determine T if OA is less than a quarter of OB, since the circles then fail to intersect.

The point T is the same as that found by the first construction for the mean proportional between $2OA$ and $\frac{1}{2}OB$, and requires that the former length should be the greater of the two. C. O. TUCKEY.

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689. "We may notice . . . an example of the absurdity of attempting to force mathematical expressions into unmathematical language. Laplace gives a description of a certain probability in these words:

'La théorie des fonctions génératrices donne une expression très simple de cette probabilité, que l'on obtient en intégrant le produit de la différentielle de la quantité dont le résultat déduit d'un grand nombre d'observations s'écarte de la vérité, par une constante moindre que l'unité, dépendante de la nature du problème, et élevée à une puissance dont l'exposant est le rapport du carré de cet écart, au nombre des observations. L'intégrale prise entre des limites données, et divisée par la même intégrale étendue à l'infini positif et négatif, exprimera la probabilité que l'écart de la vérité est compris entre ces limites.'

A student familiar with the *Théorie Analytique des Probabilités* itself might not find it easy to say what formula Laplace has in view; it must be

$$\sqrt{\frac{K}{K^n\pi}}\int dr\,.\,e^{-\frac{Kr^2}{4K^n}}."$$

—Todhunter, *History of the Theory of Probability*, pp. 498-9 [per Miss E. J. Ternouth].

690. Pascal's theorem can be obtained, for a circle, *by projection*, without projecting the circle into anything but a circle. . . . If the point of view of Apollonius had been projection—the cone a *means*, not an *end*—he would have had Pascal's theorem.—*De Morgan to Sir W. H. Hamilton*, March 1, 1858. (*Graves*, iii. 542.)

MATHEMATICAL NOTE.

946. [K¹. 1. b. γ; 2. d.] *Properties of a Certain Triangle.*

If L, M, N are the feet of the perpendiculars from a point P on the sides BC, CA, AB of a fixed triangle ABC, the triangle LMN possesses a number of interesting properties, some of which are here noted. If P is on the circumcircle ABC, LMN reduces to a straight line, the pedal (Simson's) line of P.

The following properties are true in the general case:

(1) $MN = PA \sin A$, $NL = PB \sin B$, $LM = PC \sin C$.

From this, by taking P on the circumcircle ABC, it is easy to deduce Ptolemy's theorem from the pedal line property, and conversely.

(2) *The area of* $\triangle LMN = \frac{1}{2}(R^2 - OP^2) \sin A \sin B \sin C$, so that

$$\triangle LMN : \triangle ABC = R^2 - OP^2 : 4R^2,$$

where O is the circumcentre of ABC.

This theorem may be proved trigonometrically.

If P is taken inside ABC so that PO makes an angle $\theta (< B)$ with BC, then

$$PL = R \cos A - OP \sin \theta,$$
$$PM = R \cos B + OP \sin (C + \theta),$$
$$PN = R \cos C - OP \sin (B - \theta).$$

Also $\triangle LMN = \frac{1}{2}\Sigma PM \,.\, PN \sin A$, and the result follows on substitution and reduction.

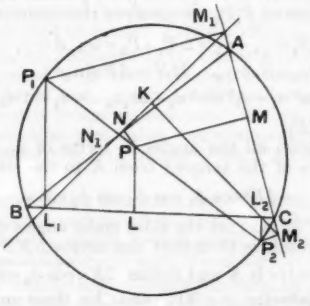
COROLLARY. *If x, y, z are the areal coordinates of P with reference to the triangle ABC, the square of the tangent from P to the circle ABC $= -\Sigma BC^2 yz$.*

This is of course a particular case of a more general theorem in areal coordinates.

(3) *If P is a variable point on a fixed diameter of the circle ABC, the circles LMN pass through a fixed point.*

If the diameter cuts the circle ABC at P_1, P_2 this point is obviously K, the intersection of the pedal lines of P_1, P_2. It is an interesting rider to prove the particular case that K lies on the nine-point circle (P being at O). It follows from the fact that the pedal lines of P_1, P_2 are perpendicular and bisect P_1H, P_2H, where H is the orthocentre of ABC.

The general theorem may be proved as follows:



If perpendiculars P_1L_1, P_2L_2; P_1M_1, P_2M_2; P_1N_1, P_2N_2 are drawn to the sides of ABC, so that $L_1M_1N_1$, $L_2M_2N_2$ are the pedal lines of P_1, P_2 and meet in K, then it will be found that $\angle KL_1L_2 = \angle AP_1P_2$, $\angle KL_2L_1 = \angle AP_2P_1$,

so that the triangles KL_1L_2, AP_1P_2 are similar. Also since by parallels $L_1L : LL_2 = P_1P : PP_2$, the triangles KLL_2, APP_2 are similar. Hence $\angle KLL_2 = \angle APP_2$.

Similarly $\angle KMM_2 = \angle BPP_2$.

Hence $\angle LKM = 360° - C - \angle APB$ (reflex in fig.).

Also $\angle LNM = 360° - C - \angle APB$, from concyclic quadrilaterals.

Therefore $\angle LKM = \angle LNM$, and thus L, M, N, K are concyclic.

This is a special case of the following still more general theorem:

(4) *If P is a variable point on any fixed line, the circles LMN cut a fixed circle orthogonally.*

If the straight line cuts the circle ABC in P_1, P_2 and the pedal lines of P_1, P_2 meet at K, it is obvious that the fixed orthogonal circle must have centre K.

Let P_1P_2 make acute angles ϕ_1, ϕ_2, ϕ_3 with BC, CA, AB. (A figure similar to the above is convenient, with P_1P_2 taken parallel to the diameter and a short distance below it.)

Take points A', B', C' on the circle ABC as follows: through A_1, the point diametrically opposite to A, draw a line parallel to P_1P_2 to cut the circle again in A'; and similarly for B', C'.

Then it will be found that $\angle KL_1L_2 = 90° - \angle AP_2P_1 = \angle A'P_1P_2$, and $\angle KL_2L_1 = \angle A'P_2P_1$. Hence the triangles KL_1L_2, $A'P_1P_2$ are similar and $KL : A'P = L_1L_2 : P_1P_2 = \cos\phi_1$. Thus $KL = A'P \sin(90° - \phi_1)$, and similarly $KM = B'P \sin(90° - \phi_2)$, $KN = C'P \sin(90° - \phi_3)$.

It will also be found that the angles of the triangle $AB'C'$ are A, $90° - \phi_1$, $90° - \phi_2$. Thus if perpendiculars are drawn from P to the sides of this triangle the lengths of the sides of the triangle formed by the feet of the perpendicular will be $= MN$, KM, KN by (1).

Hence $\triangle KMN = \frac{1}{2}(R^2 - OP^2) \sin A \cos\phi_2 \cos\phi_3$, by (2).

Also $\triangle LMN = \frac{1}{2}(R^2 - OP^2) \sin A \sin B \sin C$.

This shows that x, y, z, *the areal coordinates of K with reference to the triangle LMN, are constant for all positions of P on P_1P_2.*

$$x = \frac{\cos\phi_2 \cos\phi_3}{\sin B \sin C}.$$

The square of the tangent from K to the circle $LMN = -\Sigma MN^2 yz$, by the corollary to (2).

The reduction of this expression is a somewhat lengthy problem in trigonometry. If AP_1, AP_2; BP_1, BP_2; CP_1, CP_2 make angles α_1, α_2; β_1, β_2; γ_1, γ_2 with a fixed sense of P_1P_2 (to preserve the symmetry),

$$\alpha_2 - \alpha_1 = \beta_2 - \beta_1 = \gamma_2 - \gamma_1 = \angle P_1AP_2 = \omega; \quad A = \gamma_1 - \beta_1 = \gamma_2 - \beta_2;$$

$$\phi_1 = \beta_1 + \gamma_2 = \beta_2 + \gamma_1. \quad MN = AP \sin A;$$

$$AP^2 \sin^2\omega = x_1^2 \sin^2\alpha_1 + x_2^2 \sin^2\alpha_2 - 2x_1x_2 \sin\alpha_1 \sin\alpha_2 \cos\omega,$$

where x_1, $x_2 = P_1P$, PP_2.

Using these to express all the angles in terms of α_1, α_2, β_1, ..., it will be found that the square of the tangent from K to the circle LMN is equal to

$$-4R^2 \cos\phi_1 \cos\phi_2 \cos\phi_3 \cos\omega.$$

Write $180° - \phi_2$ for ϕ_2 (so that the sides make angles ϕ_1, ϕ_2, ϕ_3 with a fixed sense of P_1P_2); the result is then that the circles LMN all cut orthogonally a fixed circle whose centre is K and radius $2R\sqrt{\cos\phi_1 \cos\phi_2 \cos\phi_3 \cos\omega}$.

When P_1P_2 is a diameter $\omega = 90°$, and the theorem reduces to the preceding. If $\phi_1 = 90°$, the point L is fixed.

If the straight line P_1P_2 does not cut the circle the point K cannot of course be obtained by this method, but its position may be obtained from the following theorem:

(5) *If a straight line cuts the circle ABC in P_1, P_2 and the parallel diameter cuts the circle in Q_1, Q_2; the pedal lines of P_1, P_2 and of Q_1, Q_2 meet in K, K'; then if OR is the perpendicular from O to P_1P_2, K'K is equal and parallel to OR.*

This of course determines the position of K when P_1, P_2 are imaginary.

The theorem may be proved by noticing that if the perpendicular from A to BC meets the circle ABC at E, $\angle P_1KP_2 = \angle P_1EP_2$, $\angle Q_1K'Q_2 = \angle Q_1EQ_2$; and also the angle between P_1K, Q_1K' = the angle between P_2K, Q_2K'. Also if X_1, X_2; Y_1, Y_2 are the mid-points of P_1H, P_2H; Q_1H, Q_2H, then

$$X_1X_2 = \tfrac{1}{2}P_1P_2, \quad Y_1Y_2 = \tfrac{1}{2}Q_1Q_2,$$

and X_1X_2, Y_1Y_2 are parallel to P_1P_2 and their distance apart $= \frac{1}{2}OR$. X_1K, X_2K, Y_1K', Y_2K' are the pedal lines of P_1, P_2, Q_1, Q_2.

Thus if $\triangle Y_1K'Y_2$ be moved parallel to itself through a distance equal to OR, the new positions of Y_1, Y_2 will be concyclic with X_1, X_2, and K' will coincide with K.

Analytically it will be found that if O is taken as origin, the vectorial angles of A, B, C, P_1, P_2 are α, β, γ, θ_1, θ_2 and the equation of P_1P_2 is

$$x\cos\lambda + y\sin\lambda = p,$$

the coordinates of K will be

$$\tfrac{1}{2}R\{\cos\alpha + \cos\beta + \cos\gamma + \cos(\alpha+\beta+\gamma-2\lambda)\} + p\cos\lambda,$$
$$\tfrac{1}{2}R\{\sin\alpha + \sin\beta + \sin\gamma + \sin(\alpha+\beta+\gamma-2\lambda)\} + p\sin\lambda.$$

Also $\cos\omega = \cos\frac{1}{2}(\theta_2 - \theta_1) = p/R$.

H. V. MALLISON.

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691. Was there *ever* a mathematical book printed which was absolutely *free* from . . . errors ?—*Sir W. H. Hamilton to De Morgan*, Nov. 28, 1857. (*Graves*, iii. 532.)

692. " Eorum enim qui de iisdem rebus mecum aliquid ediderunt, aut solus insanio ego, aut solus non insanio ; tertium enim, nisi (quod dicat forte aliquis) insaniamus omnes."—Quoted by De Morgan from the attack of Hobbes on the geometers. (*Graves*, iii. 552.)

693. What makes the car run on after the current is cut off ? It is the force of the momentum of the power of the energy of the car.—Quoted in *Science*, July 31, 1908, p. 135.

694. If a particle start out from a given point under the simultaneous action of two impulses, it will not move in the parallelogram of the impulses, but it will move in a tautochronous, brachistochronic, plane catenary curve of a resilient character.—Quoted in *Science*, July 31, 1908, p. 135.

695. Dans sa prison, en décembre 1831, Galois écrit : " Quand la concurrence, c'est-à-dire l'*égoisme*, ne régnera plus dans les sciences, quand on *s'associera* pour étudier, au lieu d'envoyer aux Académies des paquets cachetés. . . ."

Il n'empêche que, peu de temps auparavant, " il se décide à une publication, par crainte que les géomètres plus habiles, en s'emparant du même champ, ne lui fassent perdre les fruits d'un long travail."—R. d'Adhémar, *Revue du Mois*, 1909, p. 439.

696. Le beau est une relation fixe entre des variables, une fonction comme disent les mathématiciens, quelque chose comme le cube, le carré et les puissances, lesquelles sont des choses parfaitement définies et fixes, mais par rapport à des nombres variables.—Taine, *Vie et Corr.* ii. 265.

697. J.-B. Dumas s'entretenait un jour avec Faraday d'un illustre chimiste anglais dont l'esprit de personnalité a terni la gloire. Nul plus que Faraday n'avait eu à en souffrir, mais il ne voulait pas s'en souvenir. Ramenant la pensée de Dumas sur l'œuvre du maître : " C'était, dit-il, un si grand homme." —J. Thirion, *Pascal*, Rev. des Questions Sci. 1907, p. 248. [Cf. *Gazette*, xiv. p. 430, par. 2.]

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ARTHUR BERRY.

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